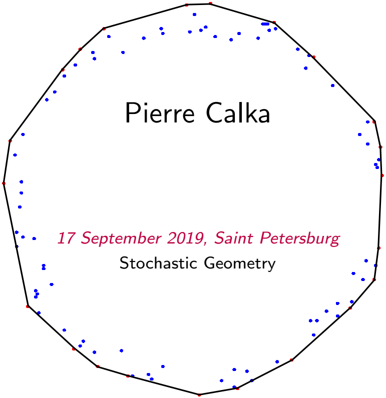
Convex hulls of perturbed random point sets



Pierre Calka

17 September 2019, Saint Petersburg

Stochastic Geometry

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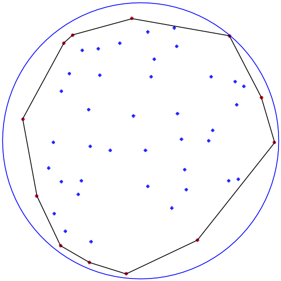
Random polytopes: a quick overview

Binomial model

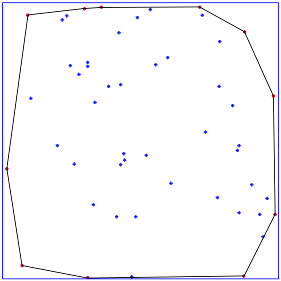
$K :=$ convex body of $\mathbb{R}^d$

$(X_k, {\scriptstyle k \in \mathbb{N}^*}) :=$ independent and uniformly distributed in K

$K_n := \mathrm{Conv}(X_1, \cdots, X_n)$, ${\scriptstyle n \geq 1}$



K_{50}, K ball



K_{50}, K square

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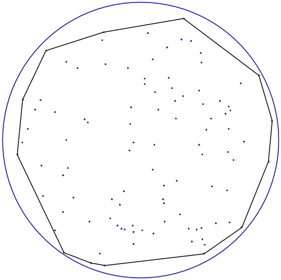
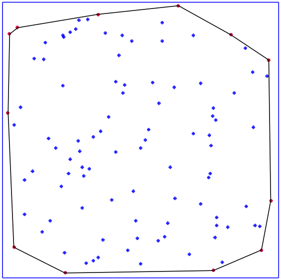


K_{100}, K ball



K_{100}, K square

Random polytopes: a quick overview

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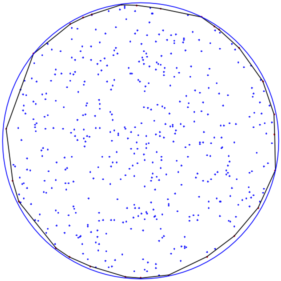
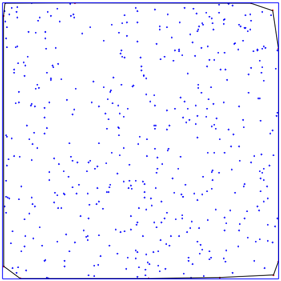


K_{500}, K ball



K_{500}, K square

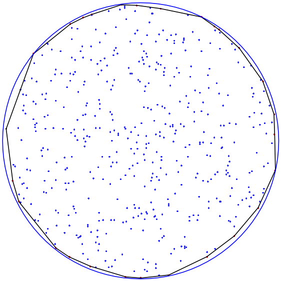
Random polytopes: a quick overview

Poisson model

$K :=$ convex body of $\mathbb{R}^d$

$\mathcal{P}_\lambda$, $\lambda > 0 :=$ Poisson point process of intensity measure $\lambda \mathrm{d}x$

$K_\lambda := \mathrm{Conv}(\mathcal{P}_\lambda \cap K)$



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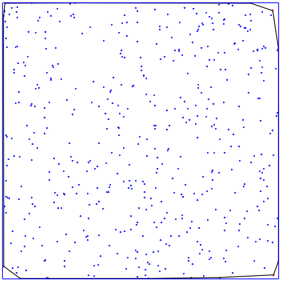


K_{500}, K square

Random polytopes: a quick overview

Expectations $f_k(\cdot) :=$ number of k-dimensional faces

K smooth $$\mathbb{E}[f_k(K_n)] \underset{n\to\infty}{\sim} c_{d,k} \int_{\partial K} \kappa_s^{\frac{1}{d+1}} \mathrm{d}s \; n^{\frac{d-1}{d+1}}$$

$\kappa_s :=$ Gauss curvature of ∂K

K polytope $$\mathbb{E}[f_k(K_n)] \underset{n\to\infty}{\sim} c'_{d,k} F(K) \; \log^{d-1}(n)$$

$F(K) :=$ number of flags of K

A. Rényi & R. Sulanke (1963), **H. Raynaud** (1970), **R. Schneider & J. Wieacker** (1978), **F. Affentranger & R. Schneider** (1992)

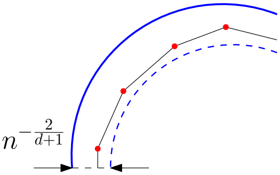
Variances

K smooth $$\mathrm{Var}[f_k(K_\lambda)] \underset{\lambda\to\infty}{\sim} c_{d,k} \int_{\partial K} \kappa_s^{\frac{1}{d+1}} \mathrm{d}s \; \lambda^{\frac{d-1}{d+1}}$$

K simple polytope $$\mathrm{Var}[f_k(K_\lambda)] \underset{\lambda\to\infty}{\sim} c'_{d,k} f_0(K) \; \log^{d-1}(\lambda)$$

Random polytopes: a quick overview

Localisation of ∂K_n, $K = \mathbb{B}^d$



With high probability

$$\partial K_n \subset \mathbb{B}^d \setminus \left(1 - c_d \left(\frac{\log(n)}{n}\right)^{\frac{2}{d+1}}\right) \mathbb{B}^d$$

Plan

Perturbed point sets

Main results

Sketch of proof

Appendix: case of the Gaussian perturbation

Joint work with **Joseph Yukich** (Lehigh University, USA)

Plan

Perturbed point sets
- Motivation: smoothed complexity
- Model of perturbed points
- Simulations

Main results

Sketch of proof

Motivation: smoothed complexity

Reference : *Smoothed analysis: why the simplex algorithm usually takes polynomial time*, D. A. Spielman & S. H. Teng (2004)

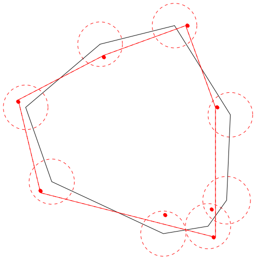
- The expectation of the number $f_0(K_n)$ of extreme points describes the mean complexity of the hull.
  $\rightsquigarrow$ In general, the data are not purely random.

- The complexity in the worst case, i.e. all points in convex position, would be n.
  $\rightsquigarrow$ This estimate of the efficiency of a construction algorithm is usually too pessimistic.

- The smoothed complexity interpolates the two complexities: we perturb an arbitrary point set and we calculate the maximum of the complexity over all possible initial configurations.

Smoothed complexity of random polytopes

Reference : *Smoothed complexity of convex hulls by witnesses and collectors*, O. Devillers, M. Glisse, X. Goaoc & R. Thomasse (2016)



Bounds for

$$\mathrm{C}omp(n,\mu) := \max_{x_1,\cdots,x_n} \mathbb{E}(f_0(\mathrm{C}onv(\{x_1+e_1,\ldots,x_n+e_n\})))$$

with $e_1,\ldots,e_n$ i.i.d. and μ-distributed
$\mu=$ uniform inside a ball or centered Gaussian distribution.

Model of perturbed points

- 2 parameters: $n \geq 1$, $\alpha \in \mathbb{R}$
- $X_1, \ldots, X_n$ i.i.d. and uniformly distributed on the unit sphere $\mathbb{S}^{d-1}$
- $e_1, \ldots, e_n$ i.i.d. and independent of the X_i,
  $e_1 := e_1(\alpha)$ uniformly distributed inside $B(0, n^\alpha)$
- $\tilde{X}_1 = X_1 + e_1, \ldots, \tilde{X}_n = X_n + e_n$

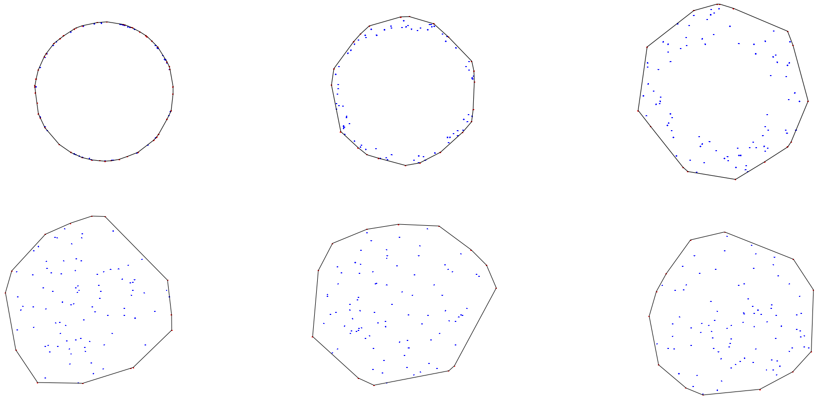
$$K_n(\alpha) := \mathrm{Conv}(\{\tilde{X}_1, \ldots, \tilde{X}_n\})$$

$K_\lambda(\alpha) :=$ corresponding Poisson model

Aim

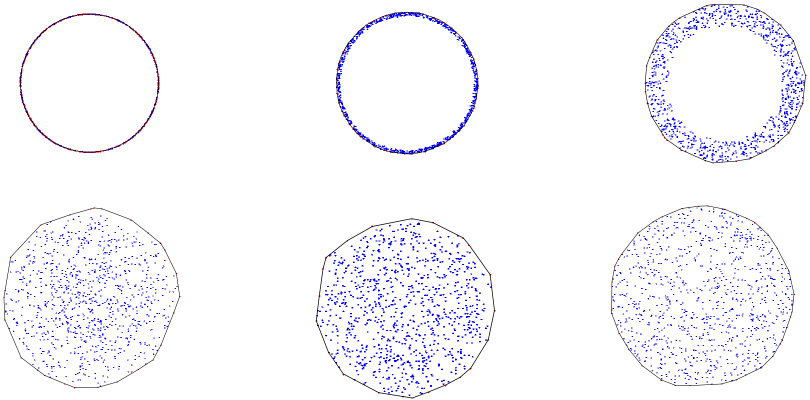
Asymptotics of $\mathbb{E}[f_k(K_n(\alpha))]$, $\mathbb{E}[f_k(K_\lambda(\alpha))]$ and $\mathrm{Var}[f_k(K_\lambda(\alpha)]$, CLT

Simulations with $n = 100$



Values of α : -1, -1/2, -1/4, 0, 1/3, 2/3

Simulations with $n = 1000$



Values of α : -1, -1/2, -1/4, 0, 1/3, 2/3

Plan

Perturbed point sets

Main results
- Statements
- Exponent for the growth rate
- Interpretation of the exponent

Sketch of proof

Main results

$$\mathbb{E}[f_k(K_n(\alpha))] \underset{n\to\infty}{\sim} m_{\infty,k}(\alpha) n^{(d-1)\beta(\alpha)},$$

$$\mathrm{Var}[f_k(K_\lambda(\alpha)] \underset{\lambda\to\infty}{\sim} v_{\infty,k}(\alpha)\lambda^{(d-1)\beta(\alpha)}.$$

$$\sup_t \left| \mathbb{P}\left(\frac{f_k(K_\lambda(\alpha)) - \mathbb{E}[f_k(K_\lambda(\alpha))]}{\sqrt{\mathrm{Var}[f_k(K_\lambda(\alpha))]}} \le t \right) - \mathbb{P}(\mathcal{N}(0,1) \le t) \right| = O\left(\frac{(\log \lambda)^{3d+1}}{\lambda^{\frac{(d-1)\beta(\alpha)}{2}}} \right)$$

Main results

$$\mathbb{E}[f_k(K_n(\alpha))] \underset{n\to\infty}{\sim} m_{\infty,k}(\alpha) n^{(d-1)\beta(\alpha)},$$

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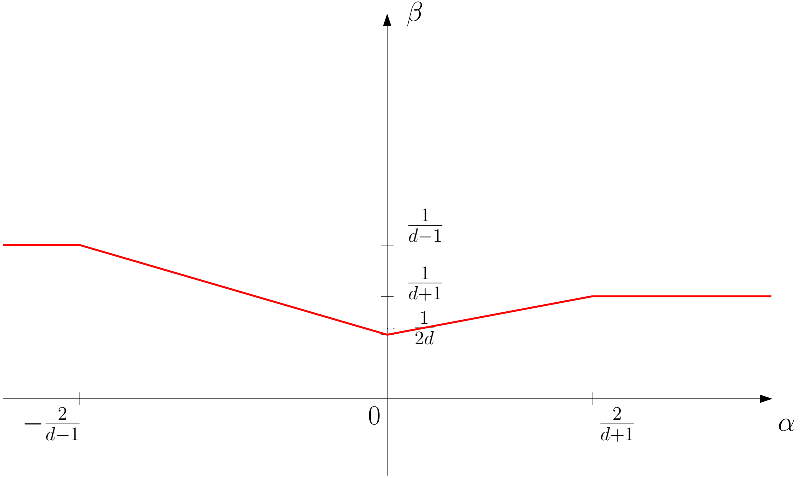
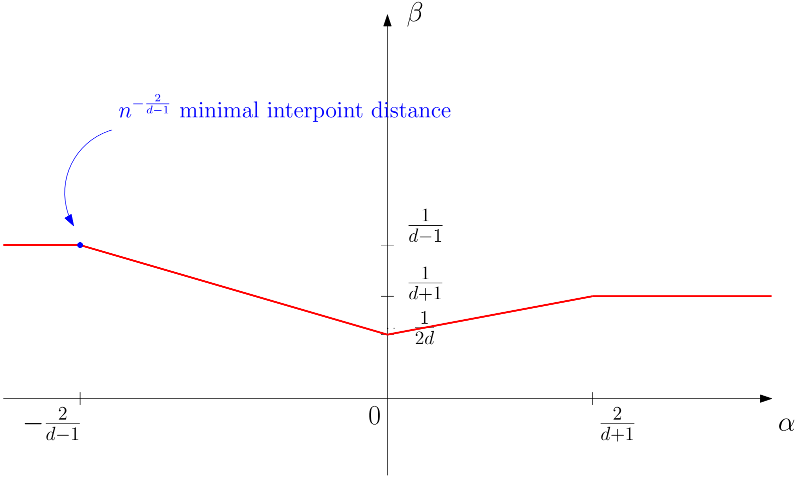
What is the exponent $\beta(\alpha)$?

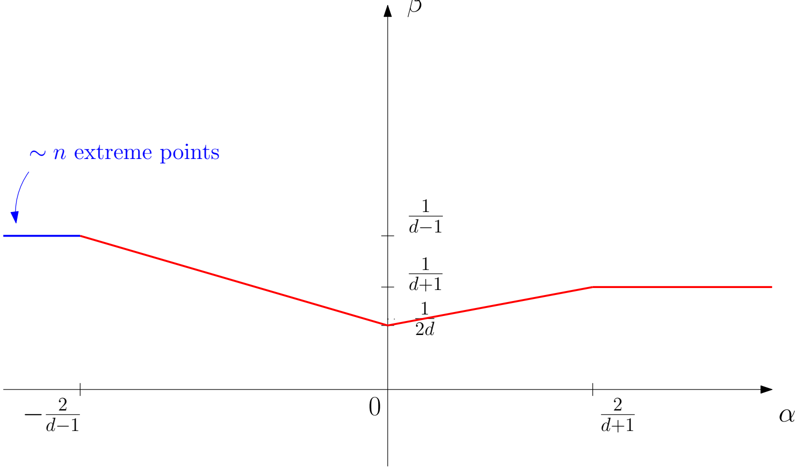
Exponent for the growth rate



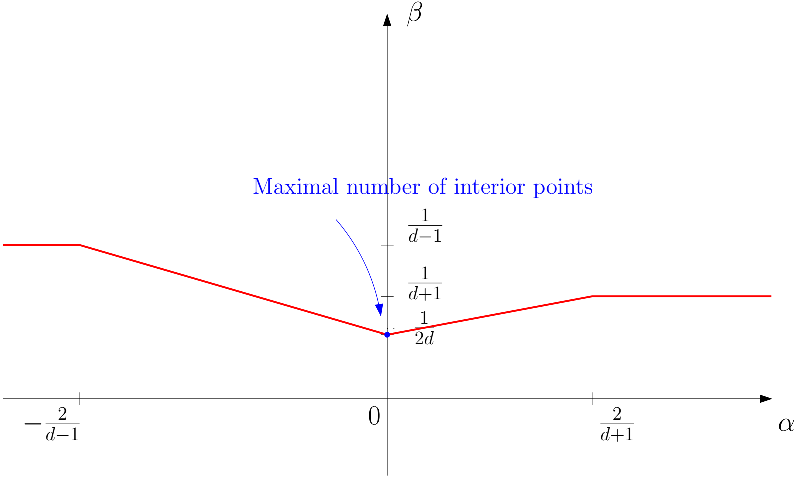
Exponent for the growth rate



Exponent for the growth rate



Exponent for the growth rate



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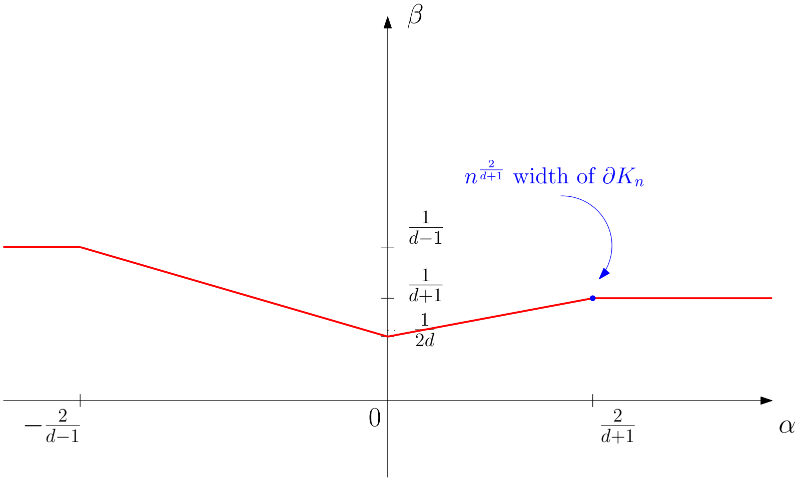
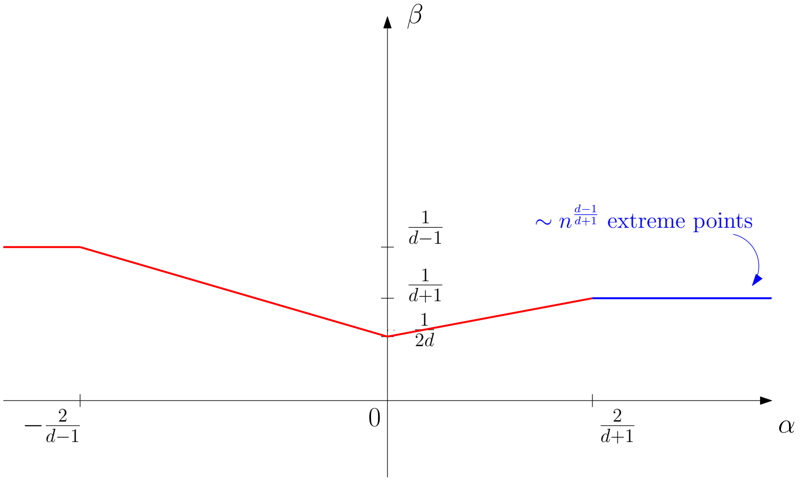


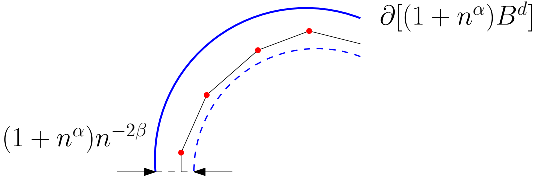
Exponent for the growth rate



Interpretation of the exponent β

$\beta := \beta(\alpha)$ is such that

- $\mathbb{E}[f_k(K_\lambda(\alpha))]$ is $\propto n^{(d-1)\beta}$,
- $\partial K_\lambda(\alpha)$ is located with high probability in an annulus of thickness $\propto (1+n^\alpha)\left(\frac{\log n}{n}\right)^{-2\beta}$.



Plan

Perturbed point sets

Main results

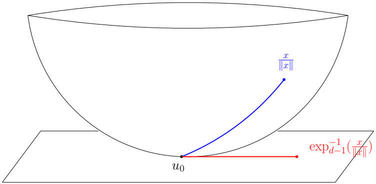
Sketch of proof

- Use of a scaling transformation
- Effect on the point process
- Effect on $K_n(\alpha)$
- Rest of the proof

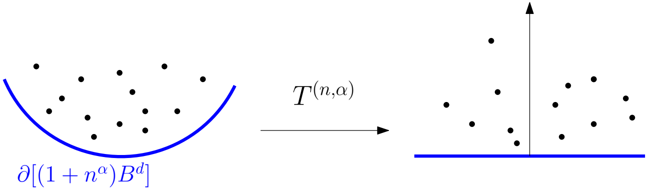
Use of a scaling transformation

$$T^{(n,\alpha)} : \left\{ \begin{array}{rcl} \mathbb{B}_d(0, 1+n^\alpha) & \longrightarrow & \mathbb{R}^{d-1} \times \mathbb{R}_+ \\ x & \mapsto & \Big((cn)^\beta \exp_{d-1}^{-1} \frac{x}{\|x\|},\ (cn)^{2\beta}(1 - \frac{\|x\|}{1+n^\alpha}) \Big). \end{array} \right.$$

$\exp_{d-1} : \mathbb{R}^{d-1} \simeq T_{u_0}\mathbb{S}^{d-1} \to \mathbb{S}^{d-1}$ exponential map at $u_0 \in \mathbb{S}^{d-1}$



Effect on the point process



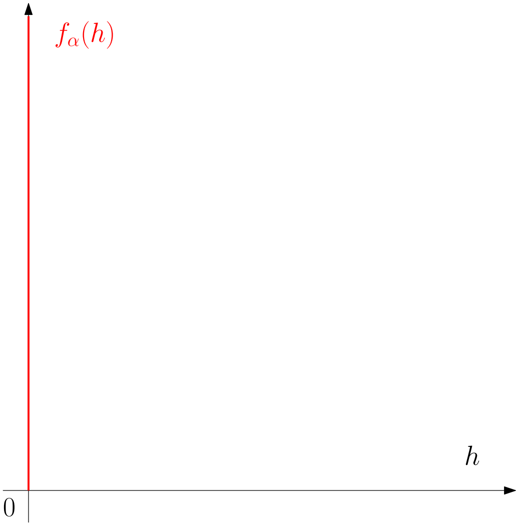
$$T^{(n,\alpha)}(\{\tilde{X}_1,\ldots,\tilde{X}_n\}) \subset \mathbb{R}^{d-1}\times\mathbb{R}_+$$

When $n\to\infty$,

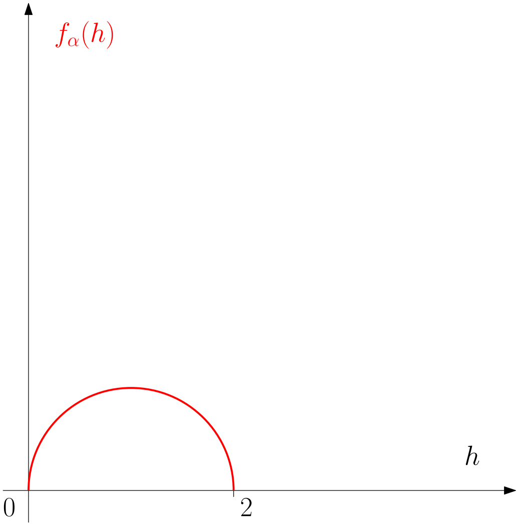
$$T^{(n,\alpha)}(\{\tilde{X}_1,\ldots,\tilde{X}_n\}) \xrightarrow{\mathcal{D}} \mathcal{P}^{(\infty,\alpha)},$$

$\mathcal{P}^{(\infty,\alpha)} :=$ Poisson point process in $\mathbb{R}^{d-1}\times\mathbb{R}_+$ invariant under horizontal translations and with explicit density $f_\alpha(h)$.

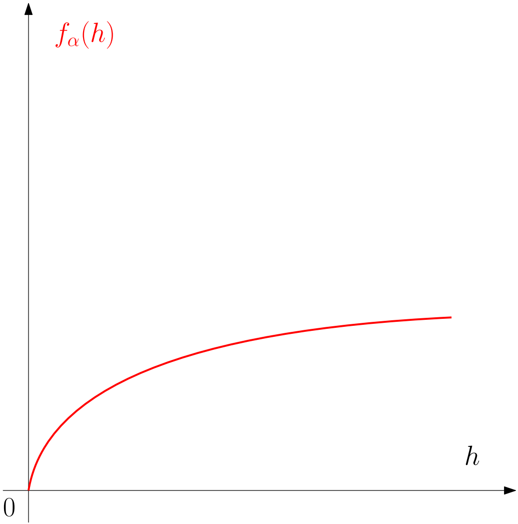
Limiting Poisson point process, $\alpha < -\frac{2}{d-1}$



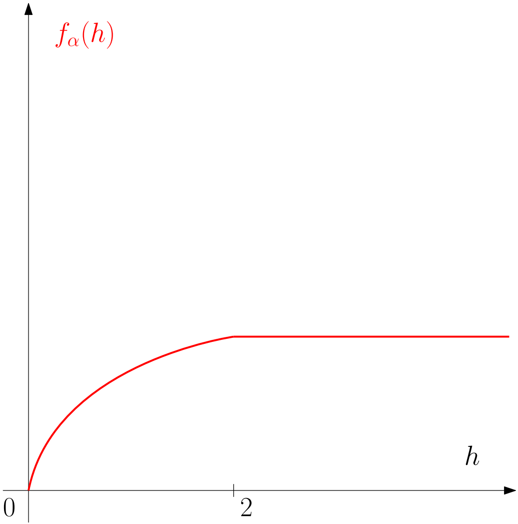
Limiting Poisson point process, $\alpha = -\frac{2}{d-1}$



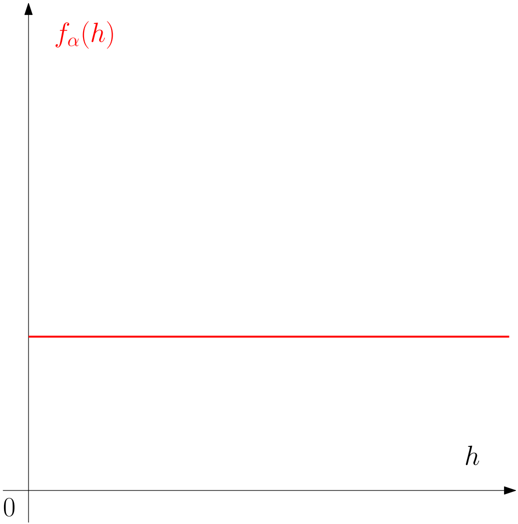
Limiting Poisson point process, $\alpha \in (-\frac{2}{d-1}, \frac{2}{d+1})$



Limiting Poisson point process, $\alpha = \frac{2}{d+1}$



Limiting Poisson point process, $\alpha > \frac{2}{d+1}$



Density of the limiting Poisson point process

$$f_\alpha(h) := \begin{cases} \mathbf{1}(h=0) & \alpha \in (-\infty, \frac{-2}{d-1}) \\ s_1(h)\cdot \mathbf{1}_{(0,2)}(h) & \alpha = \frac{-2}{d-1} \\ h^{\frac{d-1}{2}} & \alpha \in (\frac{-2}{d-1}, \frac{2}{d+1}) \\ s_2(h)\mathbf{1}_{(0,2)}(h) + \mathbf{1}_{(2,\infty)}(h) & \alpha = \frac{2}{d+1} \\ 1 & \alpha \in (\frac{2}{d+1}, \infty). \end{cases}$$

$\omega_d :=$ volume of the unit ball $\mathbb{B}^d$

$s_1(h) :=$ area of $B_d(u_0, 1) \cap (\mathbb{R}^{d-1} \times \{h\})$ with $u_0 = (0, \ldots, 0, 1)$

$s_2(h) :=$ area of the spherical cap $B_d(u_0, 1) \cap (\mathbb{R}^{d-1} \times [0, h])$

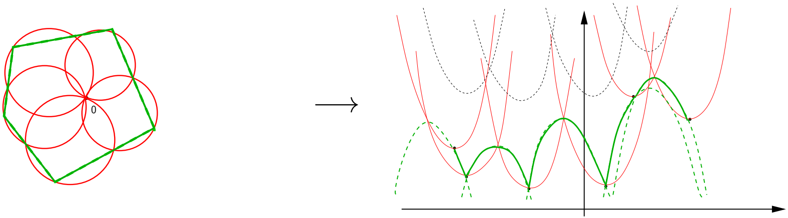
Explanation of the convergence of the point process

$$A_n(v,r,h_1,h_2) := {T^{(n,\alpha)}}^{-1}(B_{d-1}(v,r)\times[h_1,h_2])$$
$$p_n(v,r,h_1,h_2) := \mathbb{P}[\tilde{X}_1 \in A_n(v,r,h_1,h_2)]$$

$$\begin{aligned}
p_n(v,r,h_1,h_2) &= \mathbb{E}[\mathbb{P}[\tilde{X}_1 \in A_n(v,r,h_1,h_2)|X_1]] \\
&= \mathbb{E}_{X_1}\left[\frac{\mathrm{Vol}(B_d(X_1,n^\alpha)\cap A_n(v,r,h_1,h_2))}{\mathrm{Vol}(B_d(X_1,n^\alpha))}\right] \\
&= \frac{1}{\omega_d n^{d\alpha}}\int_{A_n(v,r,h_1,h_2)} \mathbb{P}(X_1 \in B_d(x,n^\alpha))\mathrm{d}x \\
&= \frac{1}{\omega_d n^{d\alpha}}\int_{A_n(v,r,h_1,h_2)} \frac{\mathcal{H}^{d-1}(B_d(x,n^\alpha)\cap\mathbb{S}^{d-1})}{d\omega_d}\mathrm{d}x
\end{aligned}$$

- Change of variables $(v,h) = T^{(n,\alpha)}(x)$
- Explicit calculation of $\mathcal{H}^{d-1}(B_d(x,n^\alpha)\cap\mathbb{S}^{d-1})$ with $|x| = (1+n^\alpha)(1-n^{-2\beta}h)$.

Effect of the scaling transformation on $K_n(\alpha)$



Initial model	Rescaled model
Perturbed points	Poisson point process in $\mathbb{R}^{d-1} \times \mathbb{R}$
Half-space	translate of $\Pi^{\downarrow}$
Boundary of $K_n(\alpha)$	Parabolic hull process
k-face of $K_n(\alpha)$	Parabolic k-face

$$\Pi^{\uparrow} := \{(v,h) \in \mathbb{R}^{d-1} \times \mathbb{R} : h \geq \frac{\|v\|^2}{2}\}, \quad \Pi^{\downarrow} := \{(v,h) \in \mathbb{R}^{d-1} \times \mathbb{R} : h \leq -\frac{\|v\|^2}{2}\}$$

Rest of the proof

- Rewriting of the functional

$$f_k(K_n(\alpha)) = \sum_{i=1}^{n} \frac{1}{k+1} \#(k\text{-faces of } K_n(\alpha) \text{ containing } \tilde{X}_i)$$

- Poissonization, Mecke's formula, change of variables provided by the scaling transformation

- Stabilization
$R(x)$:= minimal r such that $\#(k$-faces containing $x)$ only depends on $\{\tilde{X}_1, \ldots, \tilde{X}_n\} \cap B(x, r)$

$$\mathbb{P}(R(x) \geq t) \leq c e^{-t^{d-1}/c}$$

Plan

Perturbed point sets

Main results

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Appendix: case of the Gaussian perturbation

- Reminder on the Gaussian polytopes
- Model of the Gaussian perturbation
- Regimes for $\mathbb{E}[f_k(K_n(\alpha_1, \alpha_2)]$

Reminder on the Gaussian polytopes

Model $K_n :=$ convex hull of n i.i.d. $\mathcal{N}(0, I_d)$-distributed points

Expectations $$\mathbb{E}[f_k(K_n)] \underset{n\to\infty}{\sim} c_{d,k} \log^{\frac{d-1}{2}}(n)$$

A. Rényi & R. Sulanke (1963), **H. Raynaud** (1970), **F. Affentranger & R. Schneider** (1992), **Yu. Baryshnikov & R. Vitale** (1994)

Variances

$$\mathrm{Var}[f_k(K_n)] \underset{n\to\infty}{\sim} c_{d,k} \log^{\frac{d-1}{2}}(n)$$

Localisation

∂K_n included with high probability in an annulus around a critical radius $\sim \sqrt{2\log n}$ of thickness $\sim \frac{\log\log n}{\sqrt{\log n}}$

Model of the Gaussian perturbation

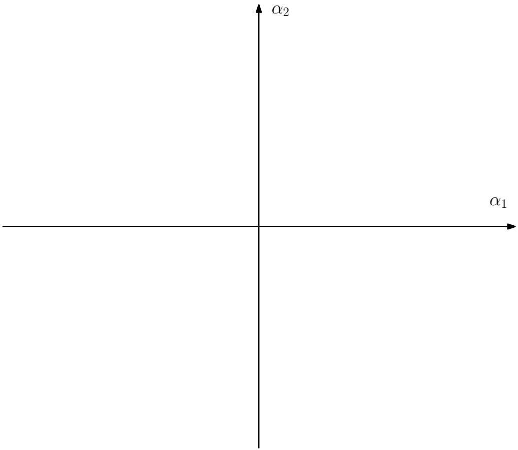
- 3 parameters: $n \geq 1$, $\alpha_1, \alpha_2 \in \mathbb{R}$
- $X_1, \ldots, X_n$ i.i.d. and uniformly distributed on the unit sphere $\mathbb{S}^{d-1}$
- $e_1, \ldots, e_n$ i.i.d. and independent of the X_i,
  $e_1 := e_1(\alpha_1, \alpha_2) \sim \mathcal{N}(0, \sigma_n^2 I_d)$ with $\sigma_n = n^{\alpha_1} (\log n)^{\alpha_2}$
- $\tilde{X}_1 = X_1 + e_1, \ldots, \tilde{X}_n = X_n + e_n$

$$K_n(\alpha_1, \alpha_2) := \mathrm{Conv}(\{\tilde{X}_1, \ldots, \tilde{X}_n\})$$

$K_\lambda(\alpha_1, \alpha_2) :=$ corresponding Poisson model

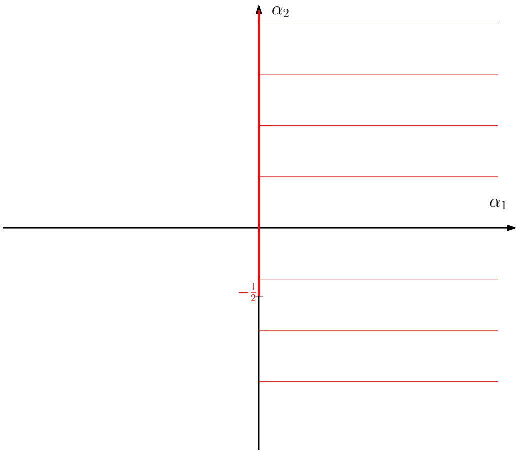
Regimes for $\mathbb{E}[f_k(K_n(\alpha_1, \alpha_2)]$

$$\mathbb{E}[f_k(K_n(\alpha_1, \alpha_2)] \sim ??$$



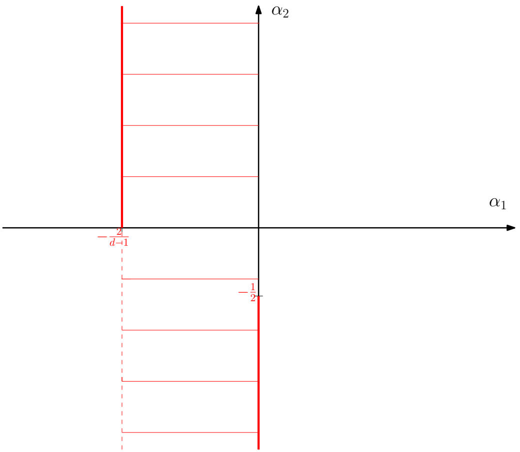
Regimes for $\mathbb{E}[f_k(K_n(\alpha_1, \alpha_2))]$

$$\mathbb{E}[f_k(K_n(\alpha_1, \alpha_2))] \sim c_{k,d} \log^{\frac{d-1}{2}}(n)$$



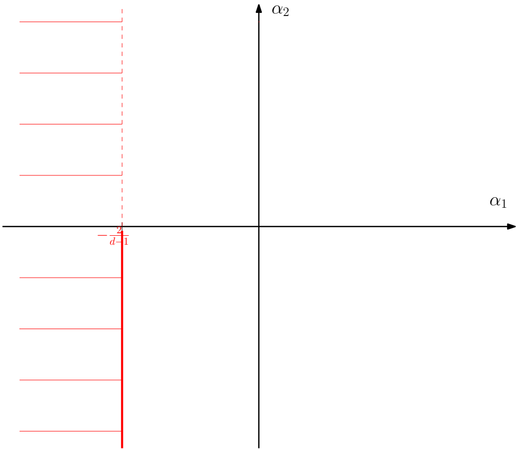
Regimes for $\mathbb{E}[f_k(K_n(\alpha_1, \alpha_2))]$

$$\mathbb{E}[f_k(K_n(\alpha_1, \alpha_2))] \sim c_{k,d}\, n^{-\frac{\alpha_1(d-1)}{2}} \log^{-\frac{\alpha_2(d-1)}{2}}(n)$$



Regimes for $\mathbb{E}[f_k(K_n(\alpha_1, \alpha_2)]$

$$\mathbb{E}[f_k(K_n(\alpha_1, \alpha_2)] \sim n$$



Thank you for your attention!