Interference Queueing Networks on Grids

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Joint work with Abishek Sankararaman and François Baccelli

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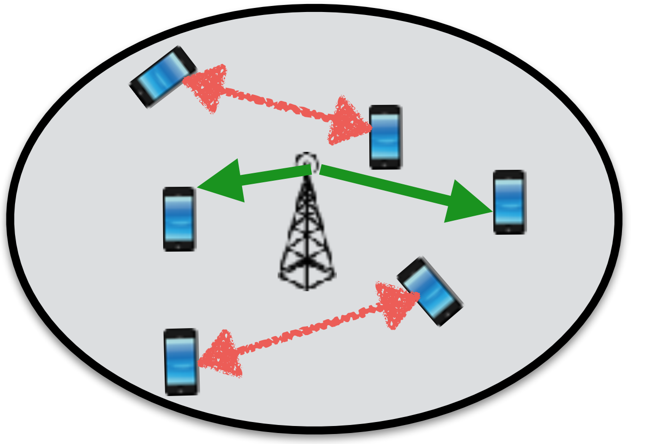
Ad-Hoc Wireless Networks

Networks without a centralized infrastructure



Examples -

1) Overlaid Device-to-Device (D2D) Networks

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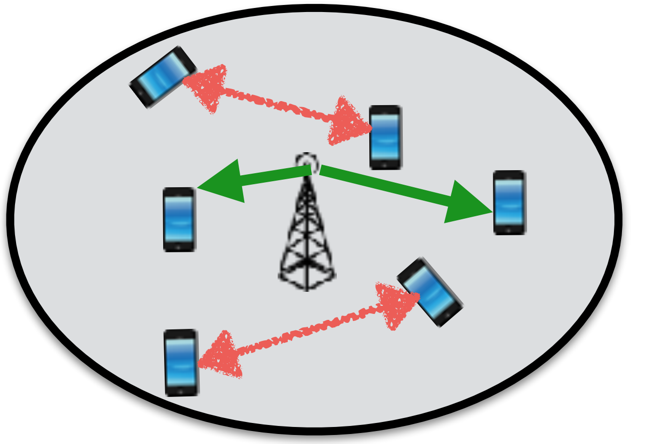
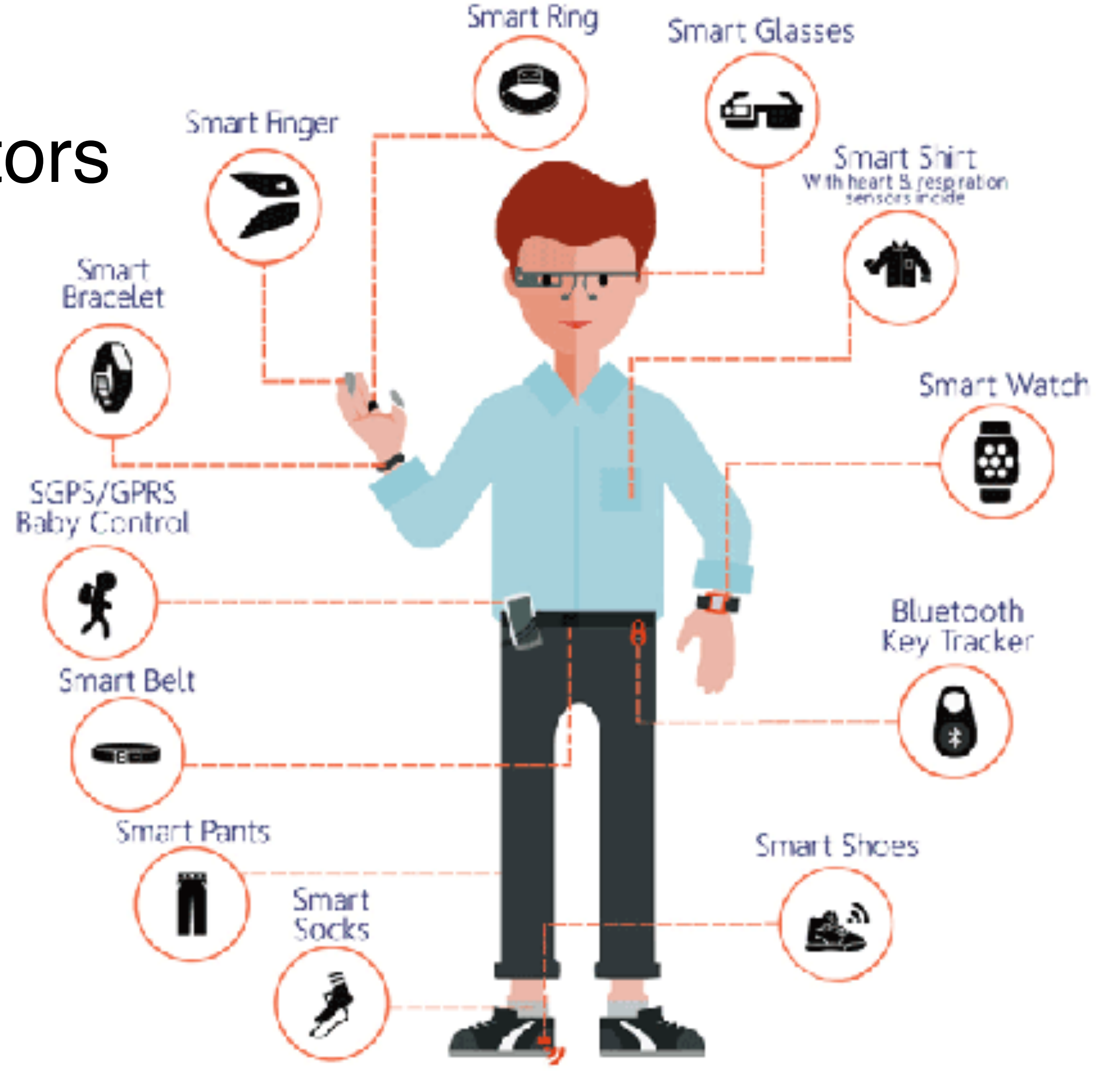


Examples -

1) Overlaid Device-to-Device (D2D) Networks

2) Internet of Things - Sensors and monitors

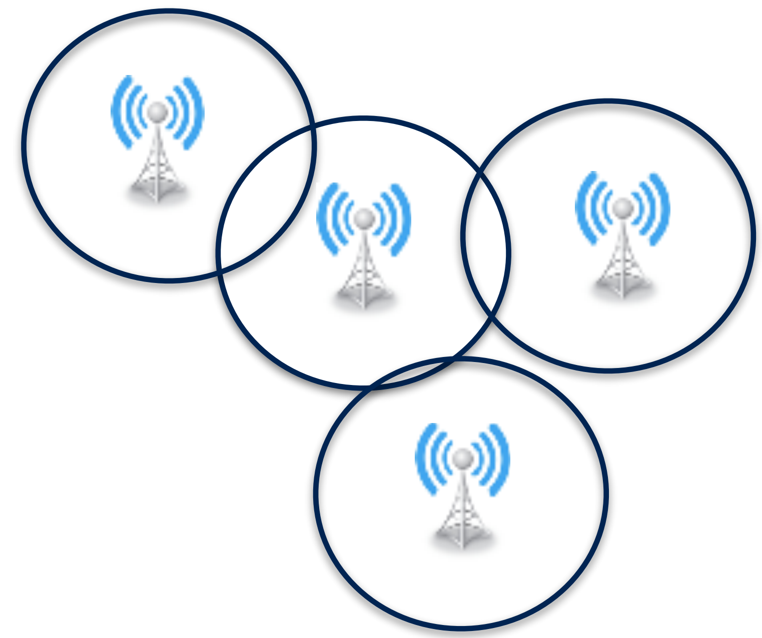


Wireless devices everywhere !

Spatio-Temporal Dynamics

Wireless Spectrum is a space-time shared resource

Spatial Component - Interference



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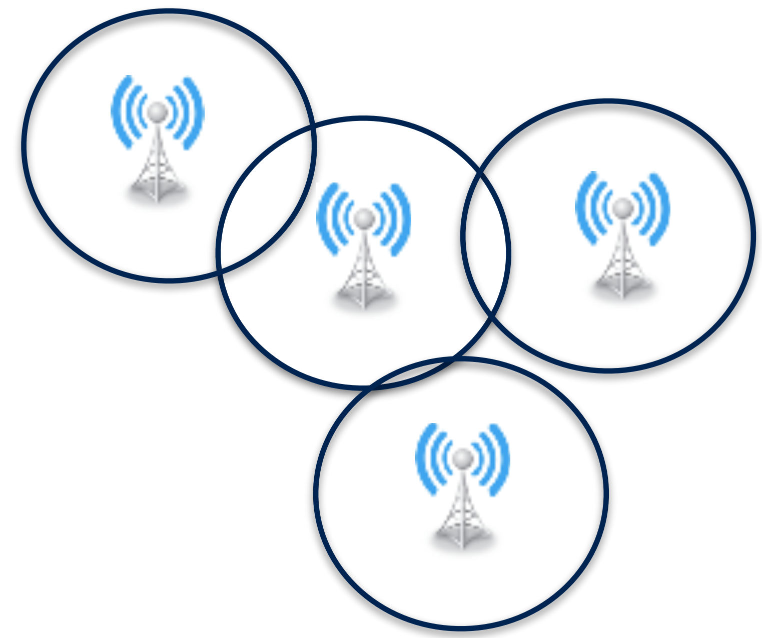
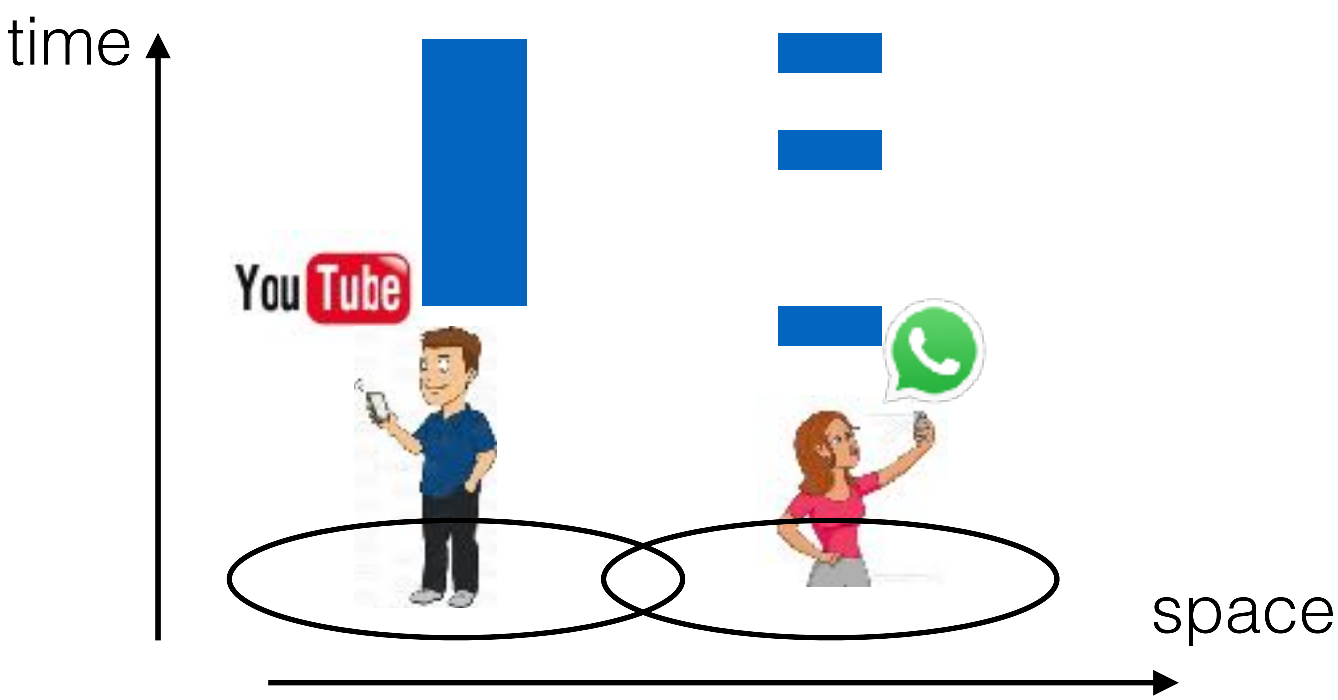


Temporal Component - Traffic Patterns



Understanding the interplay of space-time interactions is crucial for design

Prior Work

Ad-hoc networks have been studied for a long time !

However, little is understood on the spatio-temporal interactions

1. Static spatial setting

[Gupta et al. 00][Baccelli et al. 03][Andrews et al. 07][De-Veciana et al. 08][Haenggi et al. 09]

(Does not precisely capture interactions through traffic arrivals)

2. Flow-based queuing models

[Bonald et al. 06][Srikant et al. 07][Shah et al, 09][Shakkottai et al. 07][De-Veciana et al. 08]

(Does not capture precisely, the information-theoretic interactions)

We provide a framework to capture interactions in space and time

Schematic - Spatial Birth-Death Process

Protocol - *A link transmits whenever they have a file by treating interference as noise*

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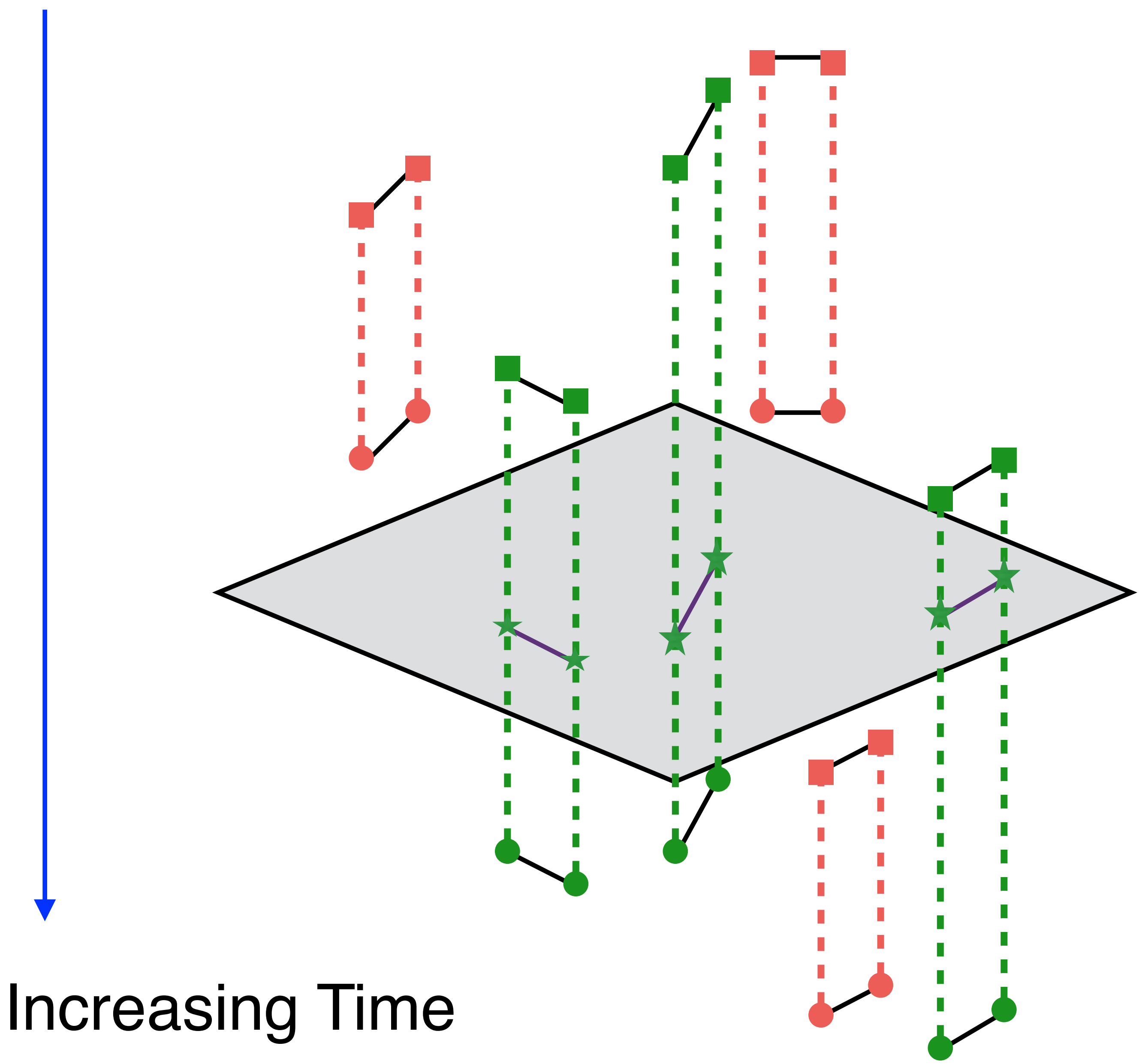
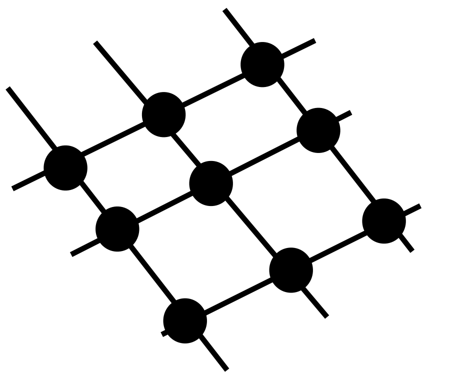


When does this protocol “work” ?

Wireless Dynamics on Grids

Protocol - *A link transmits whenever they have a file by treating interference as noise*

Discrete Space - d dimensional grid



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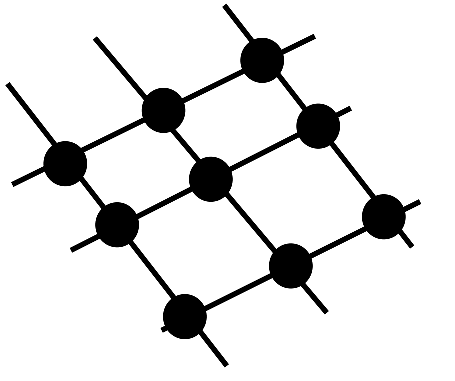

Each wireless link (Tx-Rx pair) is abstracted as a point

Links (points) 'arrive' uniformly in space and transmit

Links exit after completion of a file transfer

Wireless Dynamics on Grids

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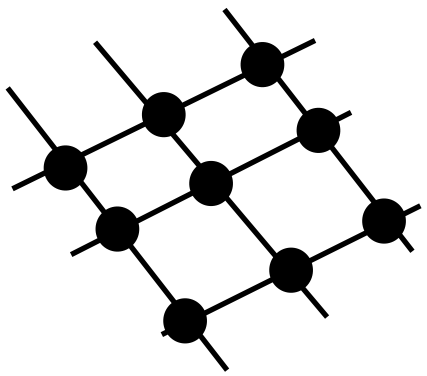

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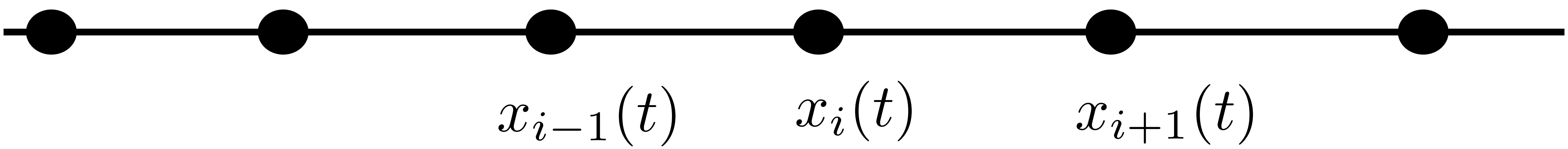
Instantaneous rate of transfer - Linearization of Shannon capacity formula

Interference as Noise

A warm up to the Model



$x_i(t) \in \mathbb{N}$ Number of links in cell $i \in \mathbb{Z}$ at time $t \geq 0$

$\{x_i(t)\}_{i \in \mathbb{Z}}$ Queue lengths at time $t \geq 0$

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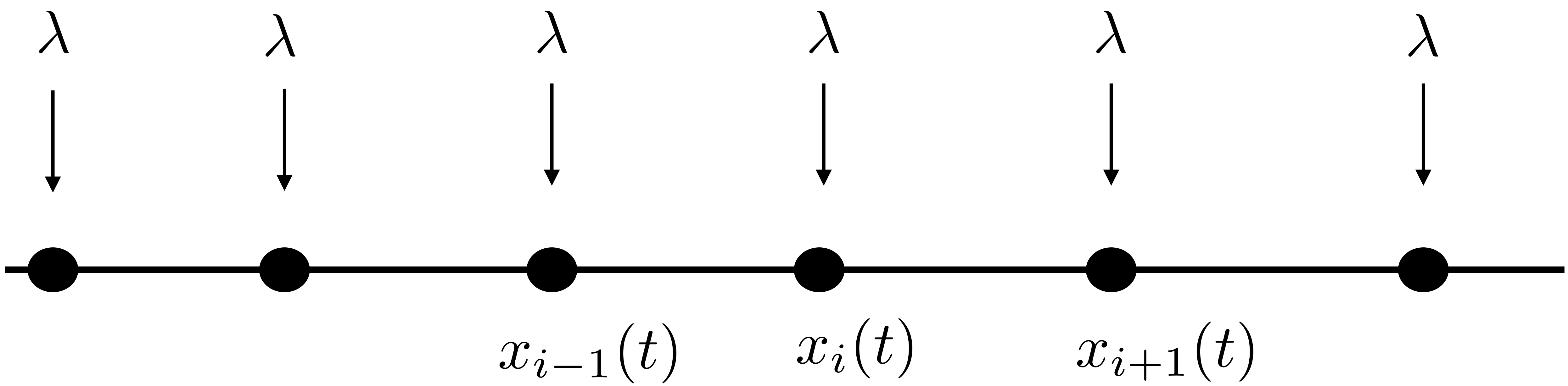

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Independent Poisson Arrivals

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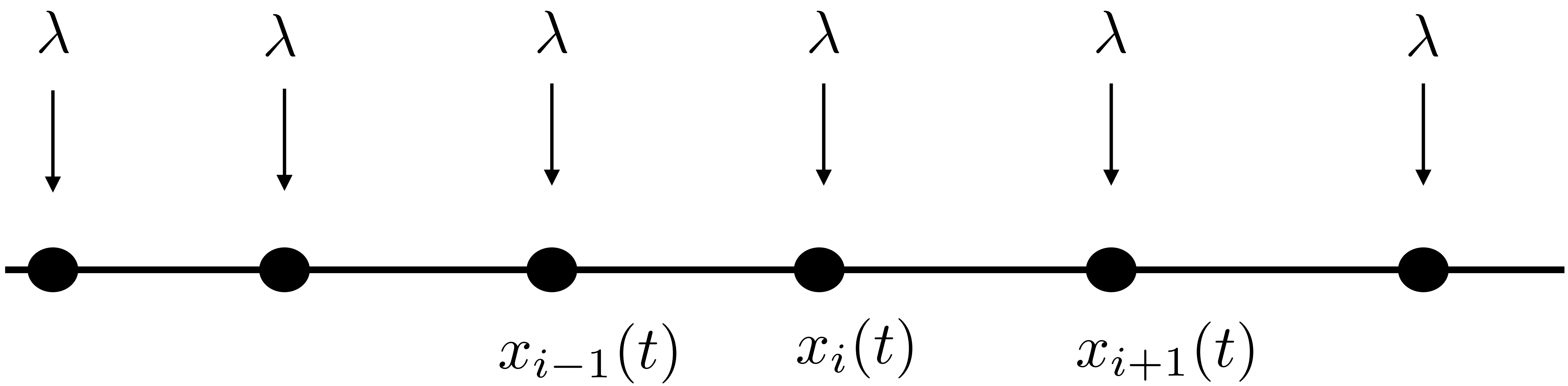

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Independent Poisson Arrivals

Rate of departure from queue $i \in \mathbb{Z}$ at time t $\dfrac{x_i(t)}{x_{i-1}(t) + x_i(t) + x_{i+1}(t)}$

If 'neighboring' queues are large, instantaneous departure rate is small.

Rate of Departure - SIR

$\{x_i(t)\}_{i \in \mathbb{Z}^d} \in \mathbb{N}^{\mathbb{Z}^d}$ Queue Lengths

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Interference Sequence $\{a_i\}_{i \in \mathbb{Z}^d}$

$a_i \geq 0 \ \forall i \in \mathbb{Z}^d$
$a_0 = 1$

$a_i = a_{-i} \ \forall i \in \mathbb{Z}^d$

$L = \sup\{||i||_\infty : a_i > 0\} < \infty$

Positivity

Symmetry

Finite Support

Interference at queue i - $\sum_{j \in \mathbb{Z}^d} a_j x_{i-j}(t)$

SIR at a customer in queue i at time t $\dfrac{1}{\sum_{j \in \mathbb{Z}^d} a_j x_{i-j}(t)}$

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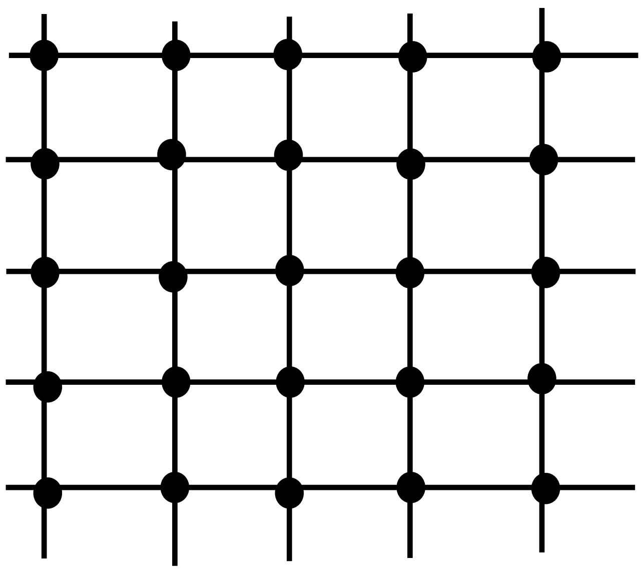
Rate of departure from any queue i at time t $\dfrac{x_i(t)}{\sum_{j\in\mathbb{Z}^d} a_j x_{i-j}(t)}$

Translation Invariant in Space

Interference Queueing Dynamics



$\{x_i(t)\}_{i \in \mathbb{Z}^d} \in \mathbb{N}^{\mathbb{Z}^d}$ Queue lengths at time $t \geq 0$

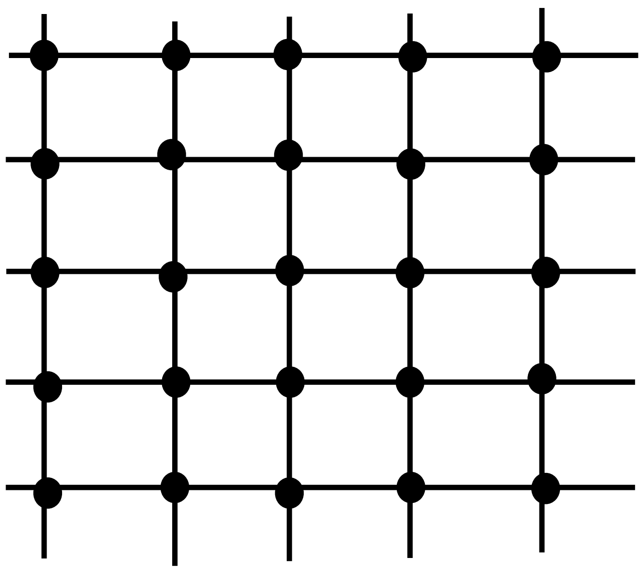
Independent rate λ Poisson arrivals

Rate of departure from queue $i \in \mathbb{Z}$ at time t $\dfrac{x_i(t)}{\sum_{j \in \mathbb{Z}^d} a_j x_{i-j}(t)}$

If 'neighboring' queues are large, instantaneous departure rate is small.

In the toy example, $a_i = 1$ if $|i| \leq 1$ and $a_i = 0$ otherwise

Interference Queueing Dynamics



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Questions -

1) For what λ and $\{a_i\}_{i \in \mathbb{Z}^d}$, is the process $\{x_i(t)\}_{i \in \mathbb{Z}^d}$ 'stable' ?

2) Characterize the steady state ??

Connection with Related Models

Few Papers discuss Infinite Queuing Networks

1) Kelbert-Kontsevich-Rybko: *On Jackson Networks on Denumerable Graphs,* 1988.

2) Foss, Chernova: *On stability of polling models with infinite number of queues*, 1996.

3) Borovkov-Korshunov-Schassberger: *Ergodicity of a polling network with an infinite number of stations*, 1999.

4) Baccelli-Foss: *Poisson Hail on a Hot Ground*, 2011.

Similarities to *Interacting Particle System* (Liggett, 1985).
However, each particle (queue) has a countable number of states.

Main Results

1. Stability

If $\lambda \sum_{j \in \mathbb{Z}^d} a_j < 1$, then system is stable

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2. Moments

Let $\{y_i\}_{i \in \mathbb{Z}^d}$ be the minimal stationary solution to the dynamics

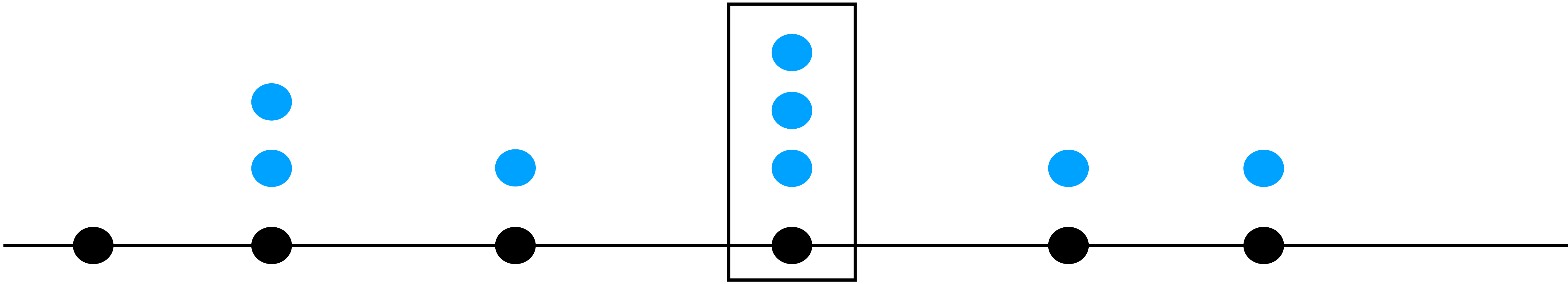
If $\lambda \sum_{j \in \mathbb{Z}^d} a_j < 1$, then $\mathbb{E}[y_0] = \frac{\lambda}{1 - \lambda \sum_{j \in \mathbb{Z}^d} a_j}$

If $\lambda \sum_{j \in \mathbb{Z}^d} a_j < \frac{2}{3}$ then $\mathbb{E}[y_0^2] < \infty$

[Shneer and Stolyar'18] established this for the entire stability range

In upcoming work by Abishek Sankararaman and Sayan Banerjee, exponential moments are shown to exist in the entire range

Intuition



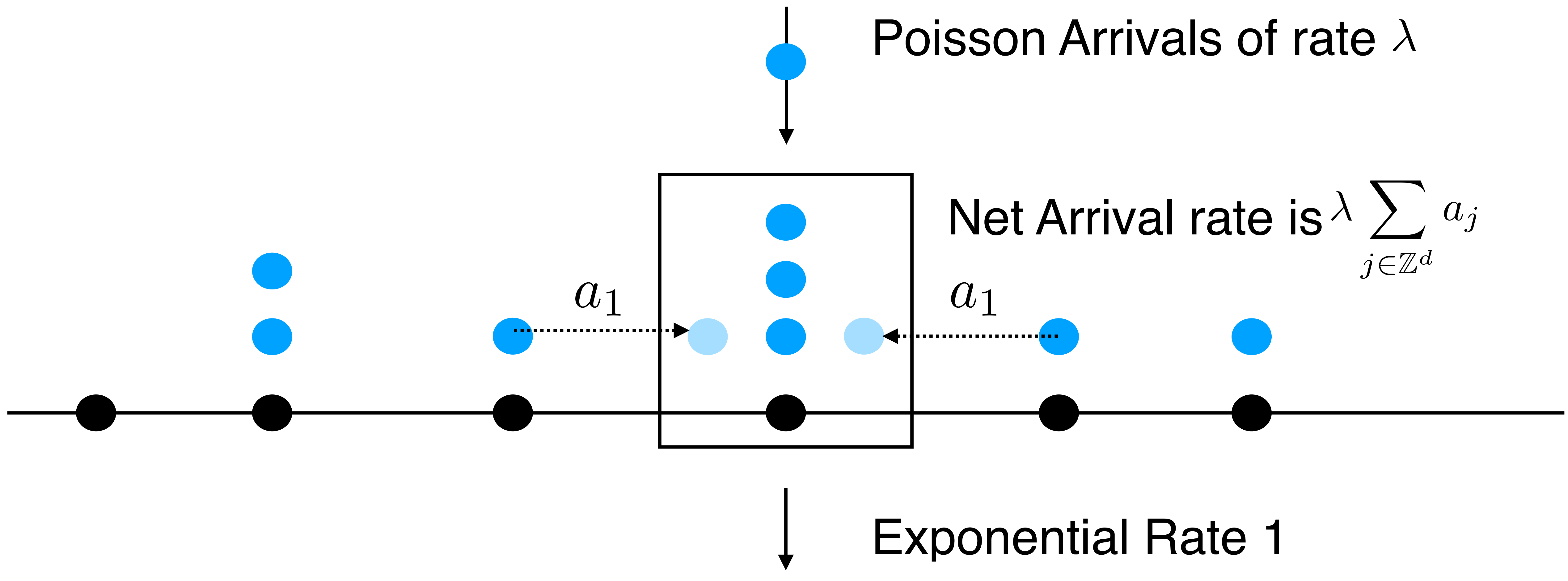
Consider any local maximum queue i, i.e. $x_i(t) = \max\{x_{i-j}(t) : a_j > 0\}$

Its instantaneous departure rate is $\dfrac{x_i(t)}{\sum_{j\in\mathbb{Z}^d} a_j x_{i-j}(t)} \geq \dfrac{1}{\sum_{j\in\mathbb{Z}^d} a_j}$

The arrival rate at every queue is λ

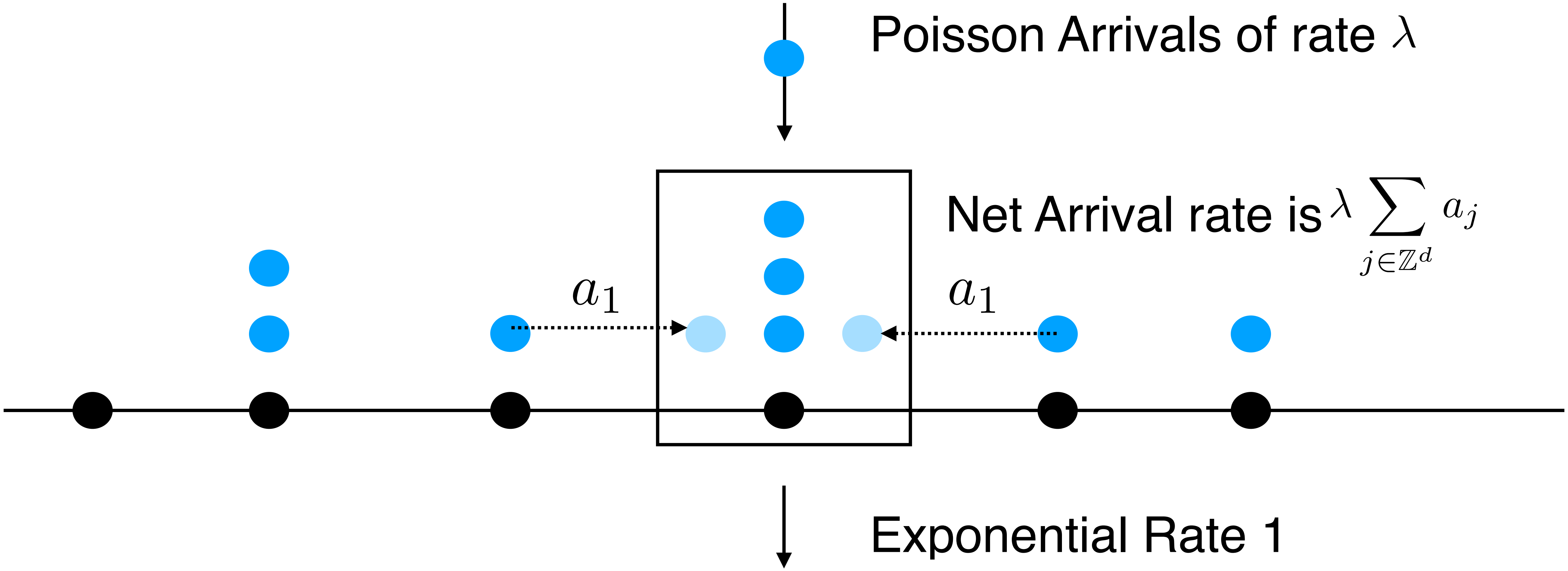
if $\lambda \sum_{j\in\mathbb{Z}^d} a_j < 1$, then this local maximum queue has negative drift

Intuition



Poisson Arrivals of rate λ
Net Arrival rate is $\lambda \sum_{j \in \mathbb{Z}^d} a_j$
a_1
a_1
Exponential Rate 1

Intuition



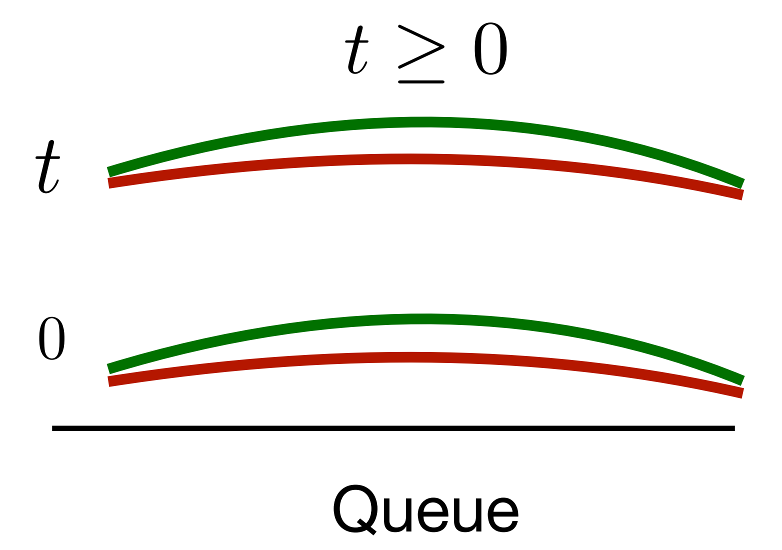
Stability - $\lambda \sum_{j \in \mathbb{Z}^d} a_j < 1$

Mean Queue Length - $\dfrac{\lambda \sum_{j \in \mathbb{Z}^d} a_j}{1 - \lambda \sum_{j \in \mathbb{Z}^d} a_j} \cdot \dfrac{1}{\sum_{j \in \mathbb{Z}^d} a_j}$

M/M/1 *Fraction of solid balls*

Monotonicity

If two initial conditions $\{x_i(0)\}_{i\in\mathbb{Z}^d}$ and $\{y_i(0)\}_{i\in\mathbb{Z}^d}$ s.t. for all $i \in \mathbb{Z}^d$ $x_i(0) \le y_i(0)$, then there exists a coupling such that almost-surely $\forall t \geq 0$, $\forall i \in \mathbb{Z}^d$ $x_i(t) \le y_i(t)$.



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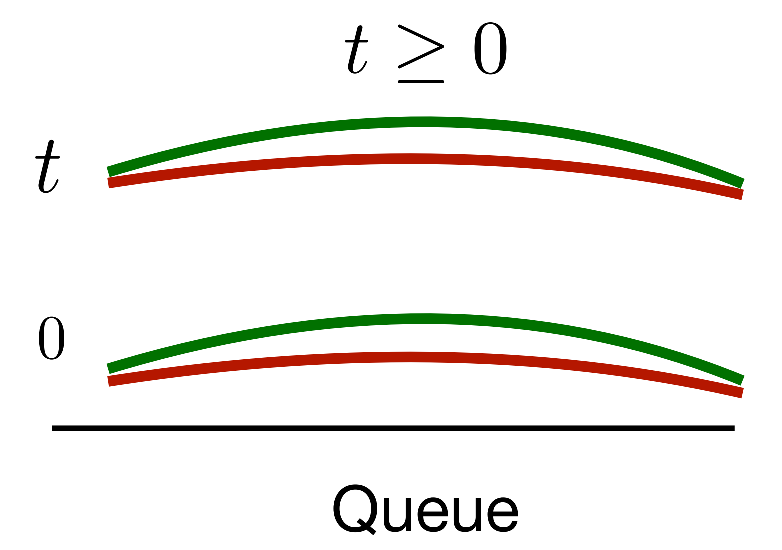

<u>Proof</u> Induction

Arrivals retain the ordering

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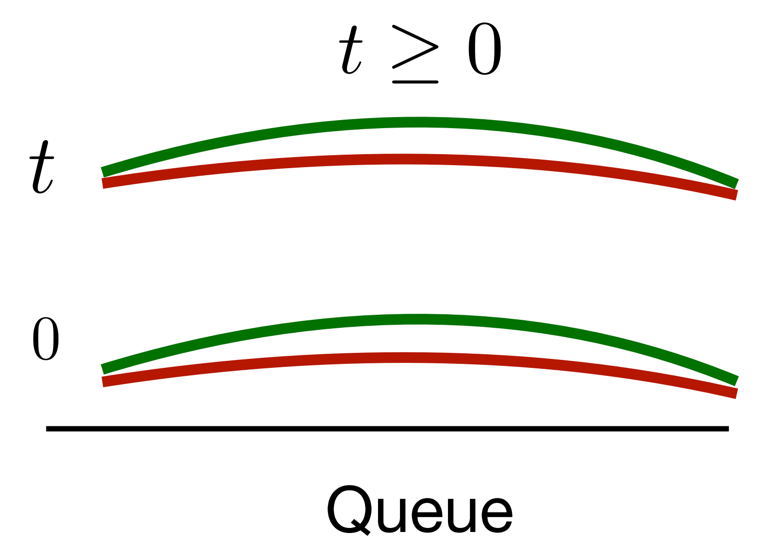
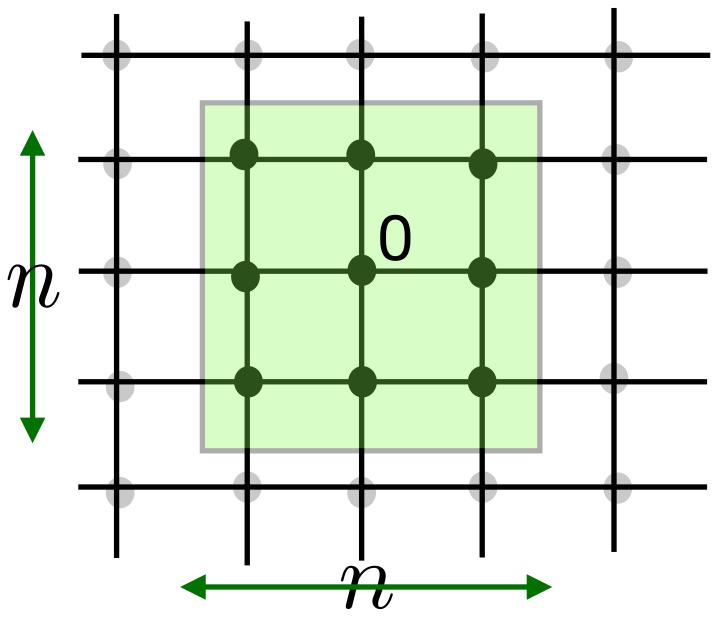


Arrivals retain the ordering

Two queues are equal - higher interference system has smaller departure

Unequal queues - Retains ordering as at-most one customer departs

Proof Steps



1. Consider a spatial truncation - finite dimensional

2. If $\lambda \sum_{j \in \mathbb{Z}^d} a_j < 1$ => Stability

   Max queue length - Lyapunov function

3. Rate Conservation Principle

$\lambda \sum_{j \in \mathbb{Z}^d} a_j < 1$ => $\mathbb{E}[y_0^{(n)}] = \frac{\lambda}{1 - \lambda \sum_{j \in \mathbb{Z}^d} a_j} - o_n(1)$ *Tightness of* $\{y_0^{(n)}\}_{n \in \mathbb{N}}$

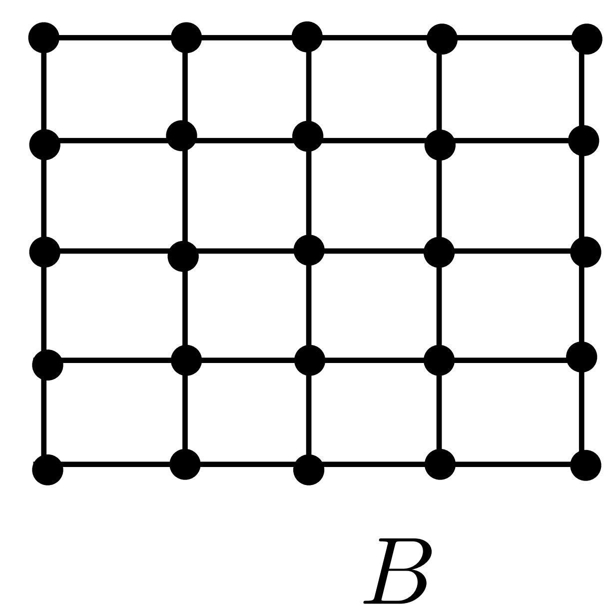
Main Proof Idea - Stability

Two systems on $B \subset \mathbb{Z}^d$ with the same dynamics.
All queues in $B^{\complement}$ are frozen without activity.

- $\{y_i(t)\}_{i \in B}$: the set B is a torus.

- $\{z_i(t)\}_{i \in B}$: the set B has boundary effects.

Interference is lower at the boundaries.



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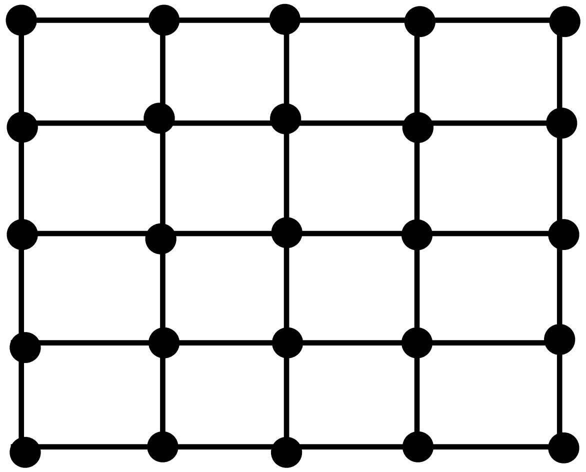
$\forall t \ \forall i \in B$

1) $x_i(t) \geq z_i(t)$

2) $y_i(t) \geq z_i(t)$

Monotonicity



B

Finite Torus System

$\{y_i(t)\}_{i \in B}$ process on a torus.

Theorem - If $\lambda \sum_{j \in \mathbb{Z}^d} a_j < 1$, then $\{y_i(t)\}_{i \in B}$ is Positive Recurrent and the stationary distribution possess exponential moments. Furthermore, the mean queue length satisfies $\mathbb{E}[y_0(t)] = \dfrac{\lambda}{1 - \lambda \sum_{j \in \mathbb{Z}^d} a_j}$

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<u>Proof Idea of Stability</u>

$$\frac{d}{dt} y_i = \lambda - \frac{y_i}{\sum_{j\in\mathbb{Z}^d} a_j y_{(i-j)/B}(t)}$$

Fluid scale equation

Consider the maximal queue $i^*(t) := \arg\max_{i\in B} y_i(t)$

$$\frac{d}{dt} y_{i^*(t)} = \lambda - \frac{y_{i^*(t)}}{\sum_{j\in\mathbb{Z}^d} a_j y_{i^*(t)-j}(t)}$$

$$\leq \lambda - \frac{1}{\sum_{j\in\mathbb{Z}^d} a_j} \quad < -\epsilon$$

This has negative drift

Can upper bound by a stable Single server queue.

Finite Torus System

Rate Conservation - "On Average what comes in is what goes out".

For Ex.

$$\lambda = \mathbb{E}\left[\frac{y_0(t)}{\sum_{j \in \mathbb{Z}^d} a_j y_{j/B}(t)} \mathbf{1}_{y_0(t)>0}\right]$$

Avg arrival rate equals avg departure rate.

<u>Key Idea:</u>

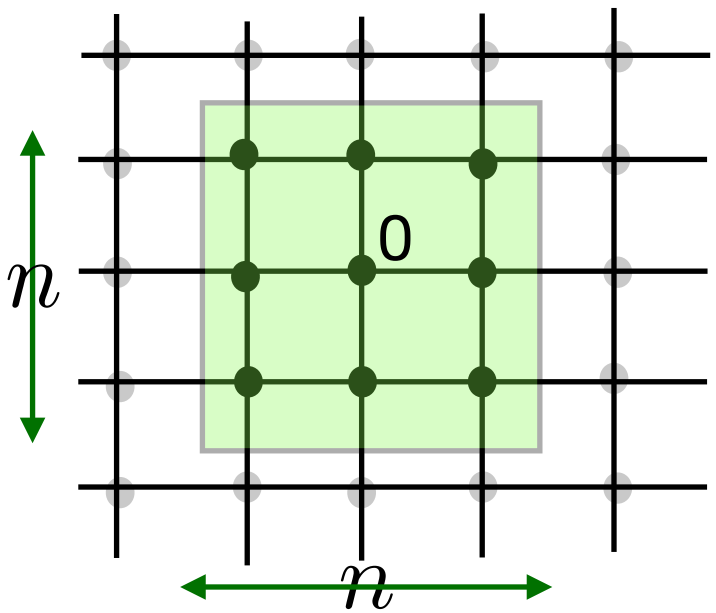
Consider $I(t) := y_0(t) \sum_{j \in \mathbb{Z}^d} a_j y_j(t)$ in stationarity and solve $\frac{d}{dt}\mathbb{E}[I(t)] = 0$

Average increase due to arrivals - $\lambda + \lambda(\sum_{j \in \mathbb{Z}^d} a_j)\mathbb{E}[y_0(t)]$

Average decrease due to departures - $\mathbb{E}[y_0(t)]$

Equating the two yields $\mathbb{E}[y_0(t)] \in \left\{\frac{\lambda}{1 - \lambda \sum_{j \in \mathbb{Z}^d} a_j}, \infty\right\}$

Proof Steps



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Tightness of $\{y_0^{(n)}\}_{n \in \mathbb{N}}$

4. Switch of limits in time and space *Coupling from the past*

5. Monotone Convergence to yield the moment formula

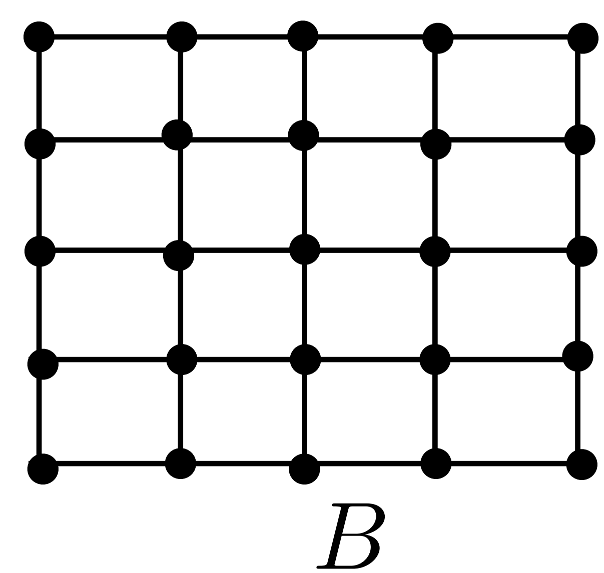
Coupling From the Past

$\{z_i(t)\}_{i\in B}$, process where the set B has boundary effects.

Monotonicity => $x_i(t) \geq z_i(t)$ and $y_i(t) \geq z_i(t)$

Thus $\mathbb{E}[z_0(t)] \leq \dfrac{\lambda}{1-\lambda\sum_{j\in\mathbb{Z}^d} a_j}$ ***Uniformly in the size of*** B

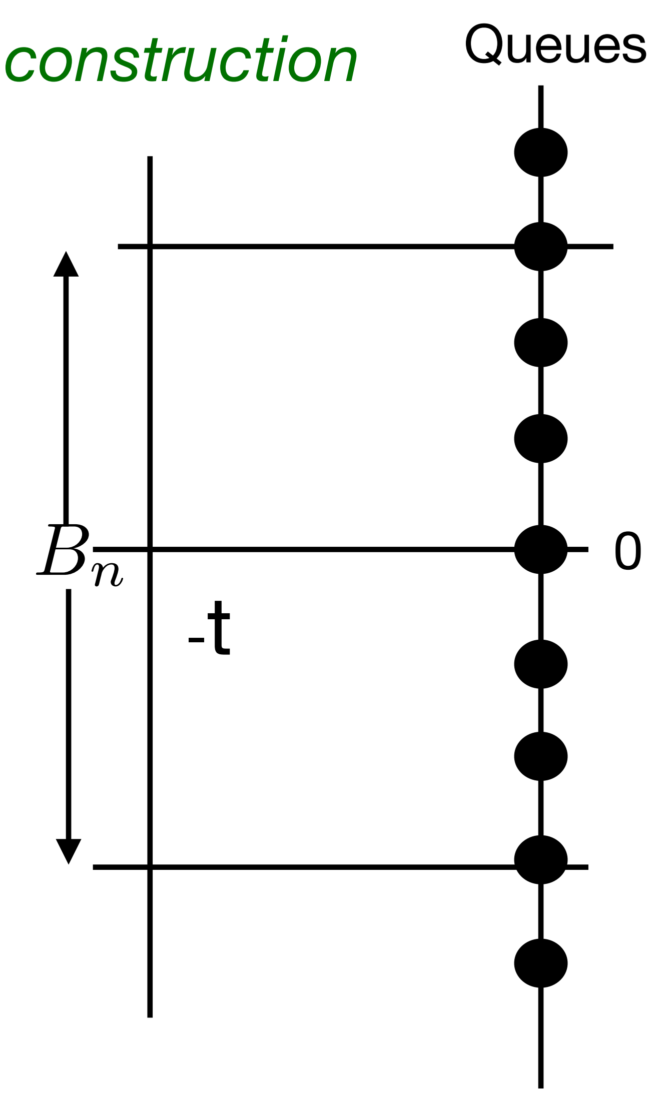
Consider $B_n \nearrow \mathbb{Z}^d$ and corresponding stationary $z_0^{(n)}(0)$



Coupling From the Past

Let $B_n \nearrow \mathbb{Z}^d$. $z_{0,t}^{(n)}(0)$ - the queue length of queue 0 at time 0, when the truncated B_n system is started empty at time -t.

Notice $\forall t \geq 0 \quad \lim_{n\to\infty} z_{0,t}^{(n)}(0) = x_{0,t}(0)$ *Corollary of the construction*



Monotonicity =>

$\lim_{t\to\infty} z_{0,t}^{(n)} := z_{0,\infty}^{(n)}$ and $\lim_{n\to\infty} z_{0,\infty}^{(n)} := z_{0,\infty}^{(\infty)}$ a.s.

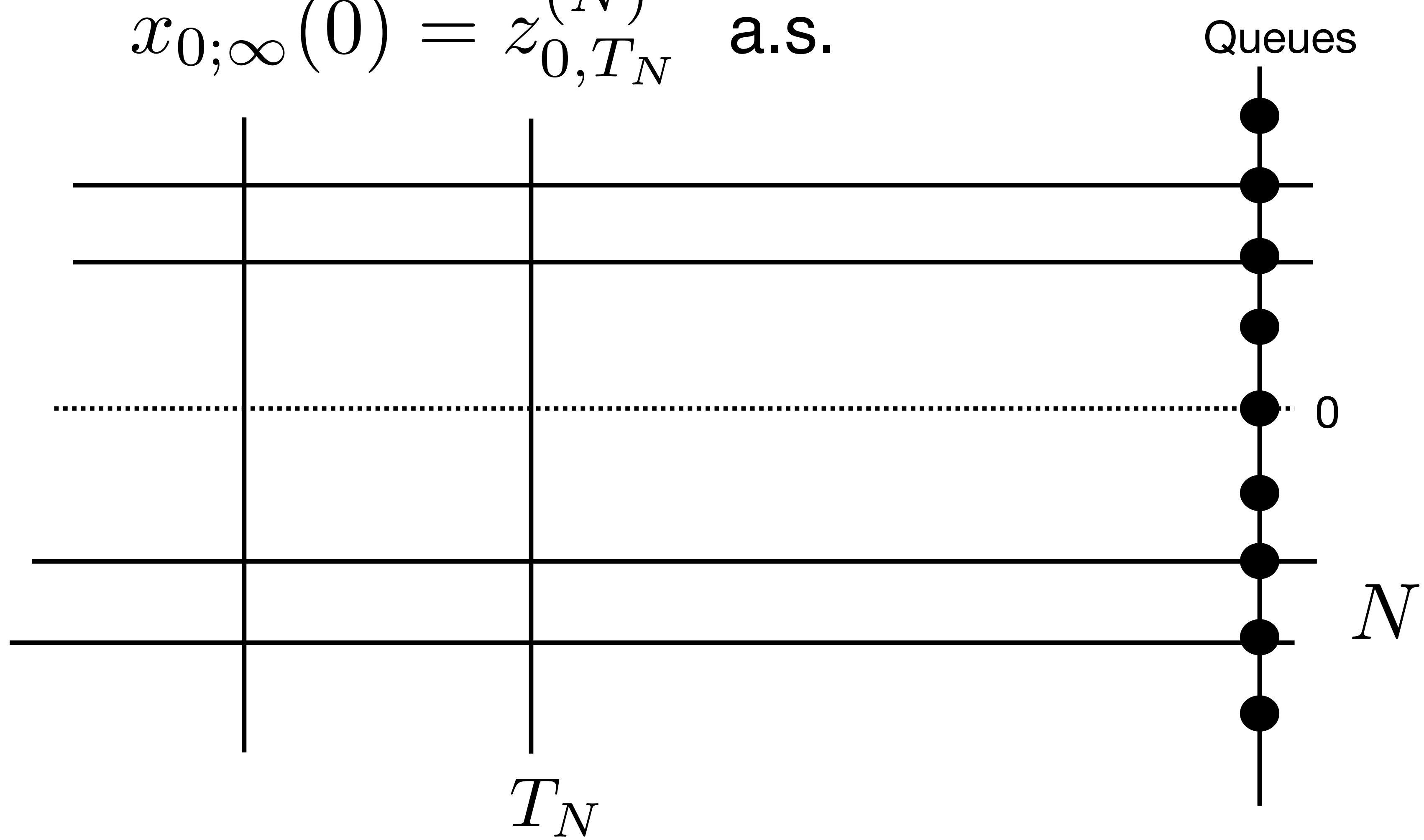
We know $\sup_{n\in\mathbb{N}} \mathbb{E}[z_{0,\infty}^{(n)}] \leq \dfrac{\lambda}{1-\lambda\sum_{j\in\mathbb{Z}^d} a_j}$

thus, $\mathbb{E}[z_{0,\infty}^{(\infty)}] < \infty$

Coupling From the Past

<u>Lemma</u> - If $\lambda \sum_{j \in \mathbb{Z}^d} a_j < 1$, then $\exists N \in \mathbb{N}$ and $\exists T_N < \infty$ random such that

$$x_{0;\infty}(0) = z^{(N)}_{0,T_N} \quad \text{a.s.}$$



We know $\sup_{n \in \mathbb{N}} \mathbb{E}[z^{(n)}_{0,\infty}] \leq \dfrac{\lambda}{1 - \lambda \sum_{j \in \mathbb{Z}^d} a_j}$. Thus $\mathbb{E}[x_{0,\infty}(0)] \leq \dfrac{\lambda}{1 - \lambda \sum_{j \in \mathbb{Z}^d} a_j}$

∎

Large Initial Conditions

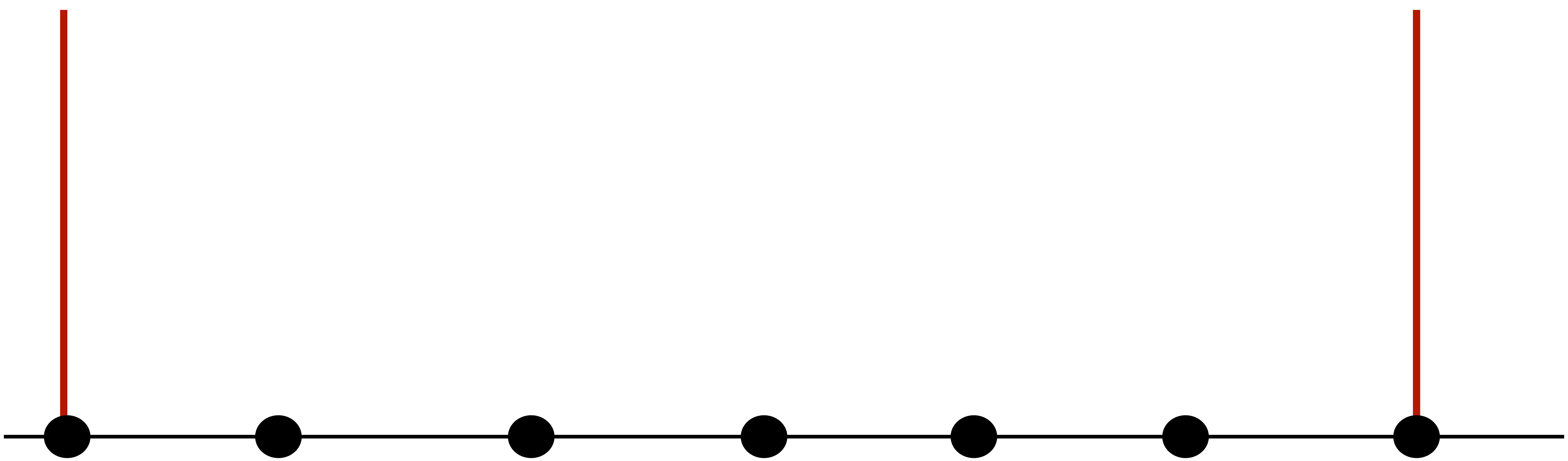
Theorem

For every λ , there exists a probability distribution on $\mathbb{N}$ such that if the initial condition is $\{x_i(0)\}_{i\in\mathbb{Z}^d}$ i.i.d. from this distribution, then $\forall i \in \mathbb{Z}^d$, $\lim_{t\to\infty} x_i(t) = \infty$ almost-surely.

Large Initial Conditions

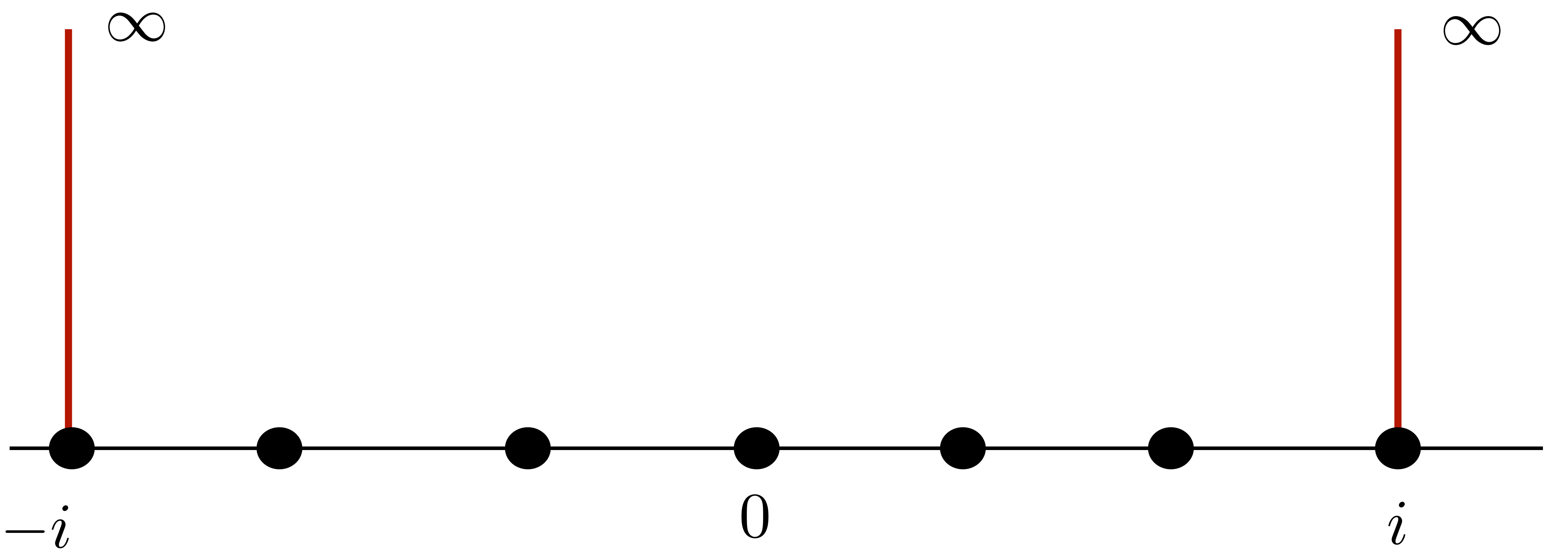
Theorem

For every λ , there exists a probability distribution on $\mathbb{N}$ such that if the initial condition is $\{x_i(0)\}_{i \in \mathbb{Z}^d}$ i.i.d. from this distribution, then $\forall i \in \mathbb{Z}^d$, $\lim_{t \to \infty} x_i(t) = \infty$ almost-surely



If “large” frozen boundary is present, then stationary queue length at 0 is also “large” with “high probability”

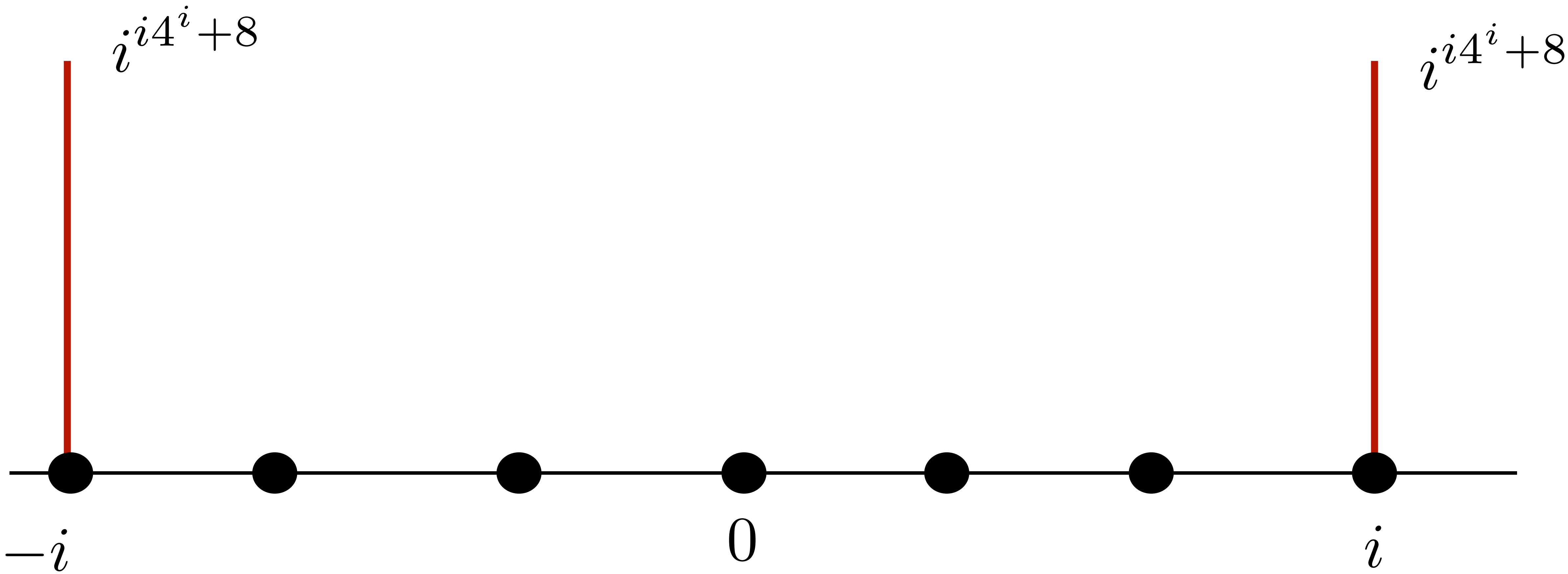
Convergence to Stationary Solutions



$$\exists (t_i)_{i \in \mathbb{N}} \text{ s.t. } t_i \to \infty \text{ s.t. } \mathbb{P}[x_0(t_i) < i] \leq i^{-4}$$

Because of the infinite barrier, all queues diverge to infinity at a linear rate

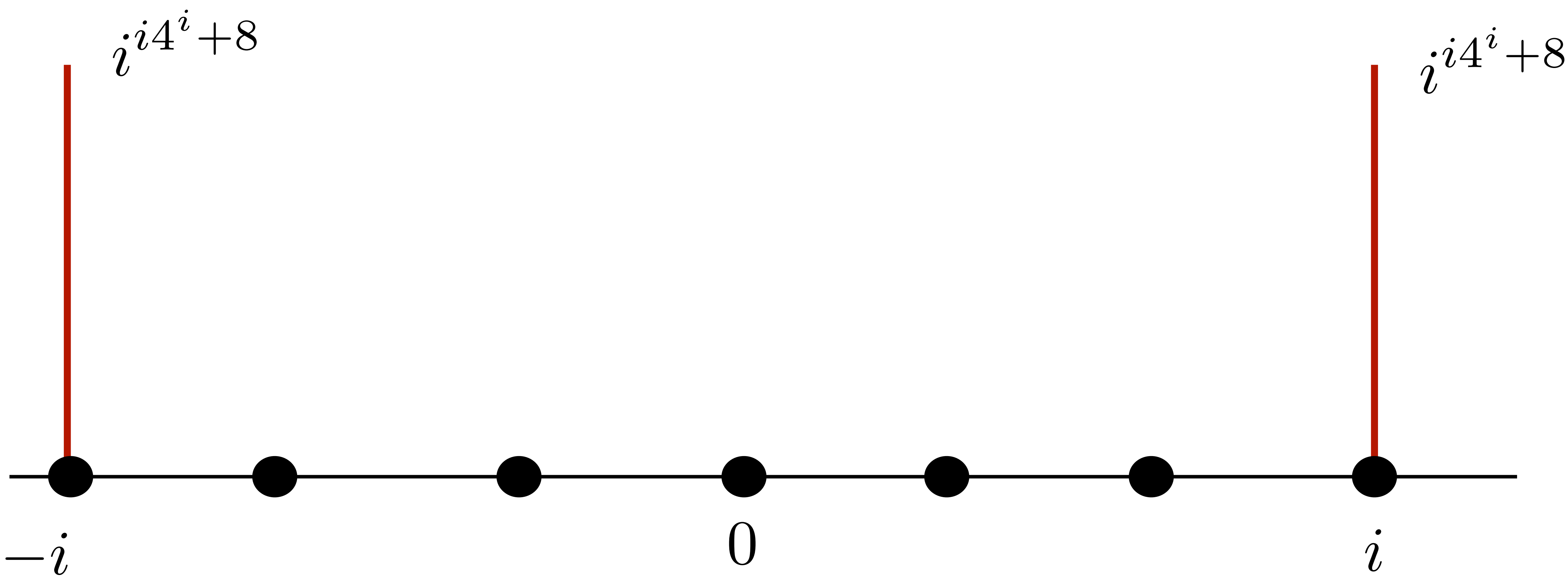
Convergence to Stationary Solutions



$$\exists (t_i)_{i\in\mathbb{N}} \text{ s.t. } t_i \to \infty \text{ s.t. } \mathbb{P}[x_0(t_i) < i] \leq i^{-3}$$

Since interested only in finite time t_i, can bring down the barrier to a finite value at a small penalty in probability

Convergence to Stationary Solutions



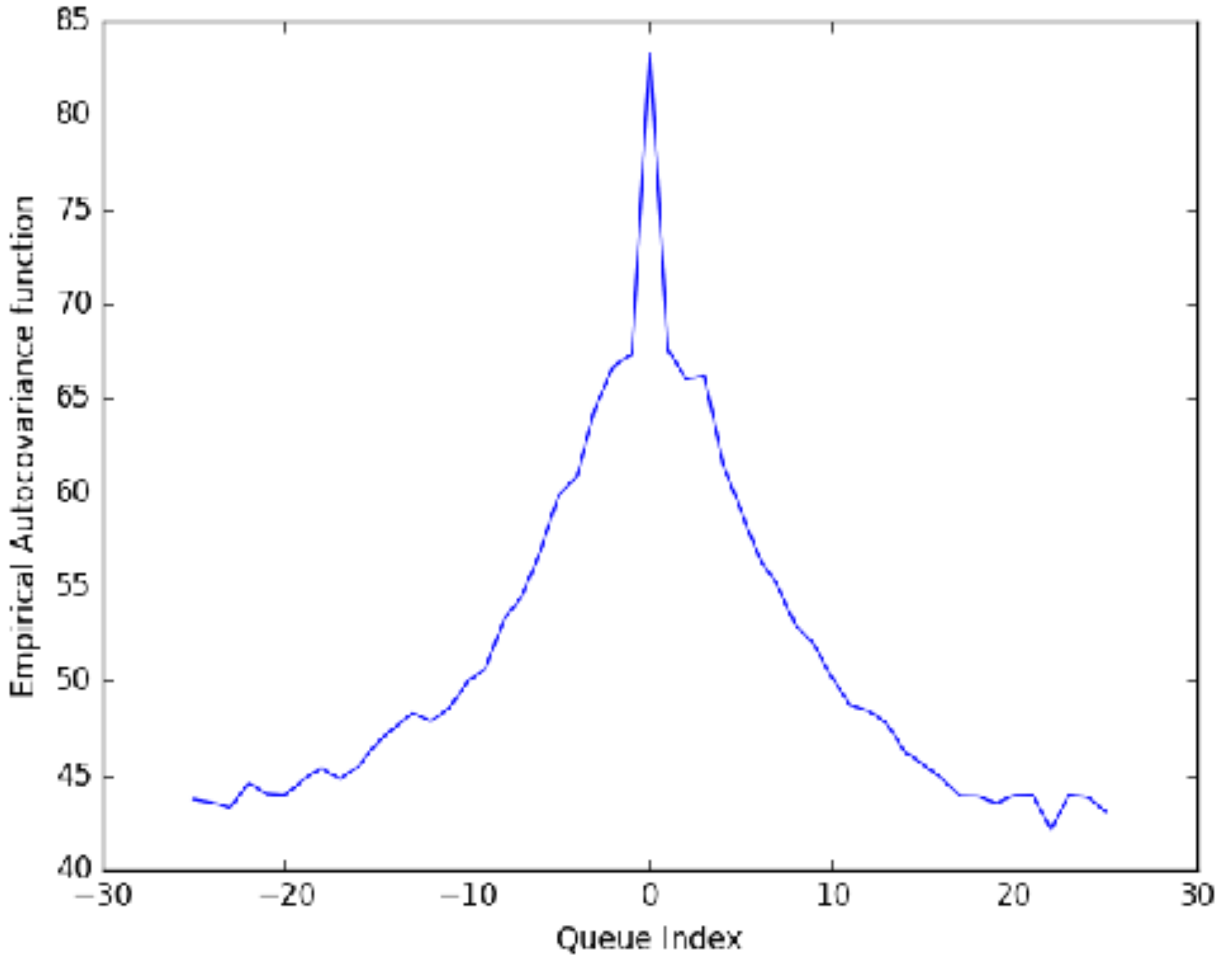
$$\exists (t_i)_{i\in\mathbb{N}} \text{ s.t. } t_i \to \infty \text{ s.t. } \mathbb{P}[x_0(t_i) < i] \le i^{-3}$$

Since interested only in finite time t_i, can bring down the barrier to a finite value at a small penalty in probability

Borel-Cantelli to conclude the proof

Open Questions

How do correlations $k \to \mathbb{E}[y_0 y_k] - (\mathbb{E}[y_0]^2)$ decay ?



d=1, n=51

$\lambda = 0.1419, \lambda_c = 1/7$

No propagation of chaos even in an infinite system !

Open Questions

Uniqueness of Stationary Solution

Existence/construction of other non-degenerate stationary solutions ?

Convergence to Stationary Solution

Do other initial conditions apart from all empty converge to a stationary limit ?

Prediction of bad outage events propagating from ‘far out’ in space

Thank You

Related Papers -

1) *Interference Queueing Networks on Grids*
A. Sankararaman, F. Baccelli and S. Foss
In Annals of Applied Probability, to appear
https://arxiv.org/abs/1710.09797

2) *Spatial Birth-Death Wireless Networks,*
A.Sankararaman and F. Baccelli
In IEEE Transactions on Information Theory, 2017
https://arxiv.org/abs/1604.07884