Spectra of sparse random graphs (and unimodularity)

Simon Coste, joint work with Justin Salez

Euler Institute - september 2019

I. Eigenvalues of random graphs: some pictures
II. Tools: Benjamini-Schramm convergence and spectral measures
III. Selected results from the litterature on spectra of sparse graphs

I

FACTS, PICTURES and QUESTIONS

on the spectra of random graphs

Examples of random graphs models

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Erdős-Rényi graphs with size $n \in \mathbb{N}$ and parameter $p \in [0, 1]$:

- $V = \{1, \ldots, n\}$.
- Put each potential edge (u, v) in E independently with probability p.

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- $\mathcal{T}_n$ = set of trees on n vertices. $|\mathcal{T}_n| = n^{n-2}$ (Cayley's formula)
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Random regular graphs with size n and parameter d:

- $\mathcal{G}_{n,d}$ = set of d-regular graphs with n vertices.
- Take G uniformly at random in $\mathcal{G}_{n,d}$.

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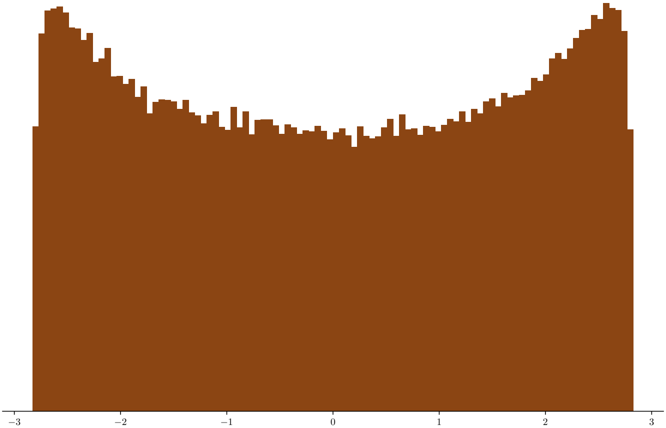
Histogram of eigenvalues of a uniform 3-regular graph on $n = 10000$ vertices (averaged over 100 realizations).



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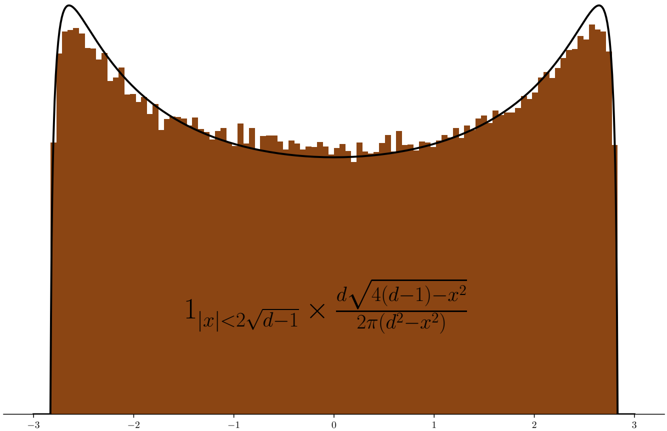

Limiting distribution = Kesten-McKay distribution

Absolutely continuous, bounded support, bounded density

Pictures: eigenvalues of Erdős-Rényi graphs, DENSE case

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$n = 10000$ vertices, $p = 1/2$; the average degree is $n/2$.
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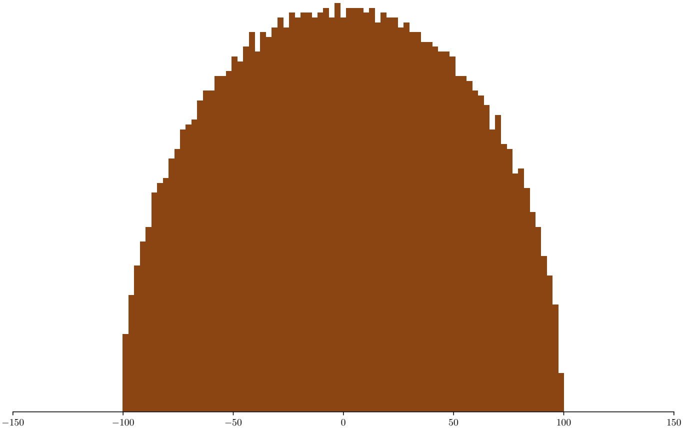

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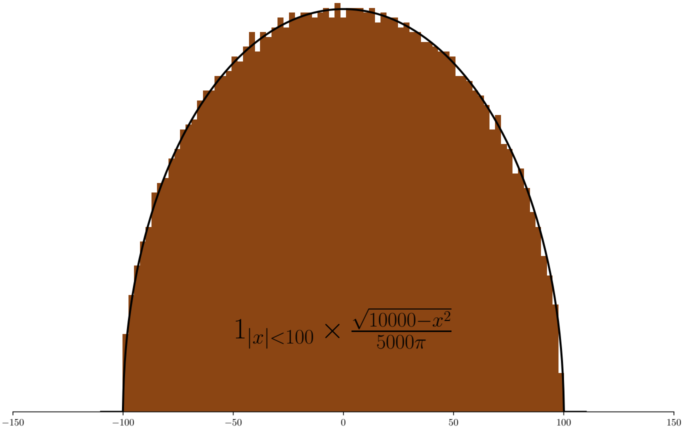

This is Wigner's semicircle distribution (rescaled).
Closed form, absolutely continuous, bounded support, bounded density.

Histogram of eigenvalues of an Erdős-Rényi graph
$n = 10000$ vertices, $p = c/n$; the average degree is c.
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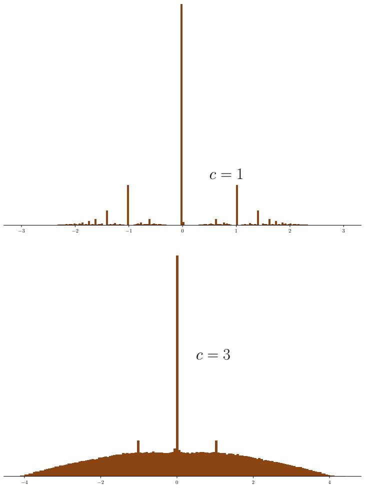

Pictures: eigenvalues of uniform trees

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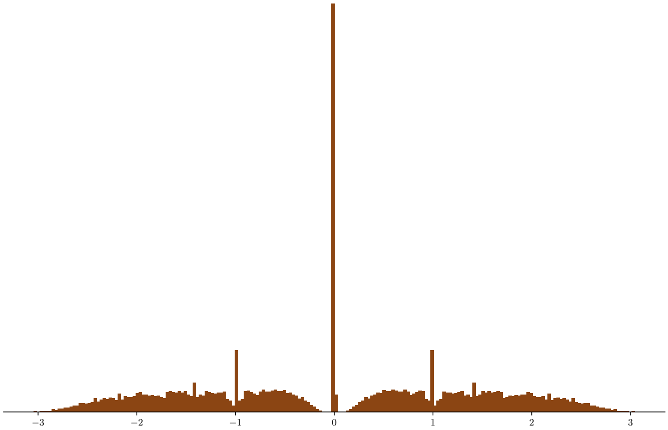
Histogram of eigenvalues of a uniform tree on $n = 10000$ vertices (averaged over 100 realizations).



II

BENJAMINI-SCHRAMM CONVERGENCE

A powerful framework for studying spectra of sparse graphs

Definition of BS convergence

$\mathcal{G}_*$ = set of rooted graphs (g, o) with a countable set of vertices
$(g, o)_t$= graph induced the ball of radius t around the root

Similarity between rooted graphs:
$\mathrm{Sim}((g, o), (g', o')) = \max\{t \in \mathbb{N} : (g, o)_t \text{ and } (g', o')_t \text{ are isomorphic}\}$

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Definition. Let G_n be a sequence of finite graphs.

- We root them uniformly at random: $o_n \sim \mathrm{Uniform}(V_n)$.
- (G_n, o_n) is now a random rooted (finite) graph.
- We say that G_n **converges in the Benjamini-Schramm sense** towards some random rooted graph (G, o) if the distribution of (Gn, o_n) converges weakly to the distribution of (G, o).

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G_n = random d-regular graph on n vertices

$$G_n \xrightarrow[n\to\infty]{\text{(BS)}} \text{d-regular tree} \tag{4}$$

One extra property: unimodularity

Benjamini-Schramm limits exhibit a special invariance property called **unimodularity** or **mass transport principle**.

Definition. A random rooted graph (G, o) is unimodular if for any measurable function $f : G \times V \times V \to [0, \infty]$ one has

$$\mathbf{E}\left[\sum_{x \in V} f(G, o, x)\right] = \mathbf{E}\left[\sum_{x \in V} f(G, x, o)\right] \tag{5}$$

Spectral continuity

Kolmogorov-Smirnov continuity [Abért, Thom, Virag, 2015]

Suppose that

$$G_n \xrightarrow[n\to\infty]{\text{(BS)}} (G, o)$$

where (G, o) is a random rooted graph with distribution ρ.

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Then there is a probability distribution μ_ρ such that

$$\sup_{t\in\mathbb{R}} |F_{\mu_{G_n}}(t) - F_{\mu_\rho}(t)| \to 0 \tag{6}$$

where F is the cumulative distribution function.

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A remark:

$$\text{Convergence (6)} \iff \begin{cases} \text{Weak convergence} \\ + \\ \text{Convergence of the atoms} \end{cases}$$

Representation of the limiting distribution

Let (G, o) be a (possibly infinite) rooted graph with bounded degree and let A be its adjacency operator on $\ell^2(V)$.
$(e_v : v \in V)$ canonical basis of $\ell^2(V)$

Herglotz theory

There is a probability measure $\mu_{(G,o)}$ such that for any $z \in \mathbb{C}_+$

$$\langle e_o, (A - z)^{-1} e_o \rangle = \int_{\mathbb{R}} \frac{1}{\lambda - z} \mu_{(G,o)}(\mathrm{d}\lambda). \tag{7}$$

- $\mu_{(G,o)}$ is called the spectral measure of G at the root.
- (7) holds if A is unbounded, but essentially self-adjoint
- If (G, o) is unimodular, then it is almost surely essentially self-adjoint.

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Representation of the limiting distribution

Suppose that $G_n \xrightarrow{(BS)} (G, o)$ with distribution ρ. Then $\mu_{G_n} \to \mu_\rho$ and

$$\mu_\rho = \mathbf{E}_\rho[\mu_{(G,o)}]. \tag{8}$$

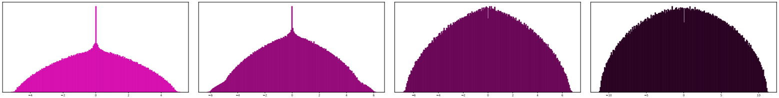
III

SOME RESULTS

on the limiting spectrum of some sparse graphs

Convergence towards semi-circle

Histograms of eigenvalues of Erdős-Rényi graphs with parameter c/n and size $n = 5000$ (average over 100 realizations):



(a) $c = 5$ (b) $c = 8$ (c) $c = 10$ (d) $c = 30$

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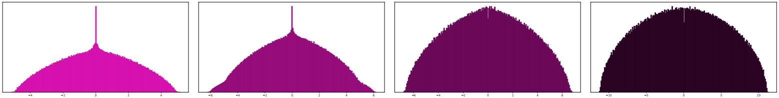

(e) $c = 5$ (f) $c = 8$ (g) $c = 10$ (h) $c = 30$

[Jung, Lee, 2017]

$$\frac{\mu_c}{\sqrt{c}} \xrightarrow[c \to \infty]{(d)} \text{Wigner semicircle distribution}$$

Atom at zero

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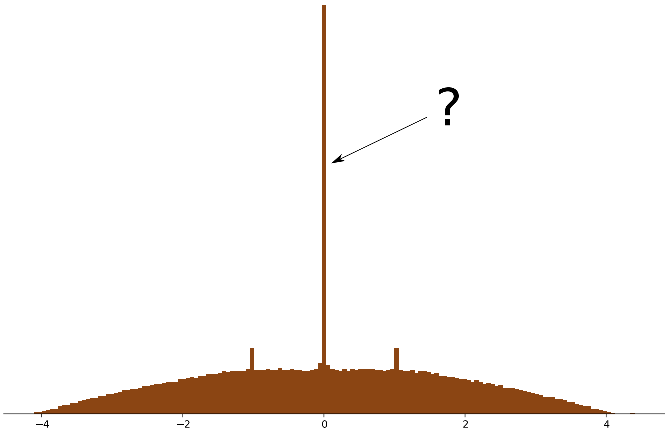
'Histogram' of μ_c with $c = 3$



Atom at zero: computation is feasible

[Bordenave, Lelarge, Salez 2015]: atom at zero for Poisson(c) GW trees

$$\mu_c(\{0\}) = \mathrm{e}^{-c\alpha} + c\alpha\mathrm{e}^{-c\alpha} + \alpha - 1$$

where α is the smallest solution of $x = \mathrm{e}^{-c\mathrm{e}^{-cx}}$ in $(0, 1)$.
$+$ generalization to any unimodular GW trees

[Bauer, Golinelli, 2000]: atom at zero for the skeleton tree

$$\mu_{\text{skel}}(\{0\}) = 2\beta - 1$$

where $\beta \approx 0.567\ldots$ is the unique solution in $(0, 1)$ of $x = \mathrm{e}^{-x}$.

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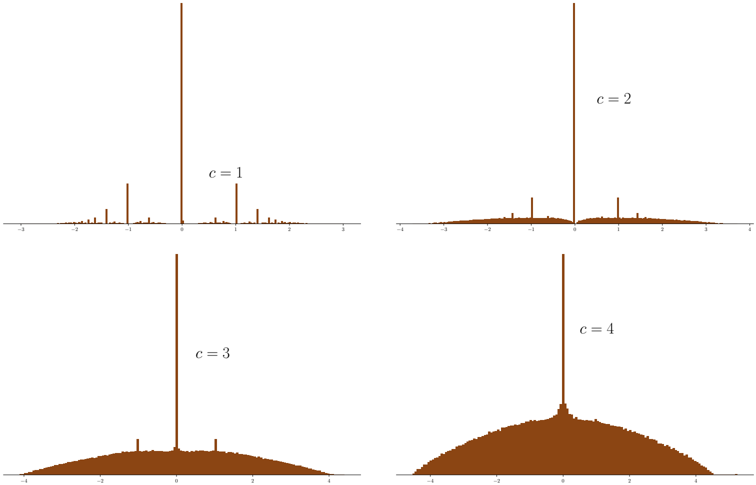

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- The proof is not constructive.
- What is proven is that the total mass of atoms is strictly smaller than 1. We cannot infer existence of an absolutely continuous part or a singular continuous part.
- The proof extends to unimodular trees with bi-infinite paths.

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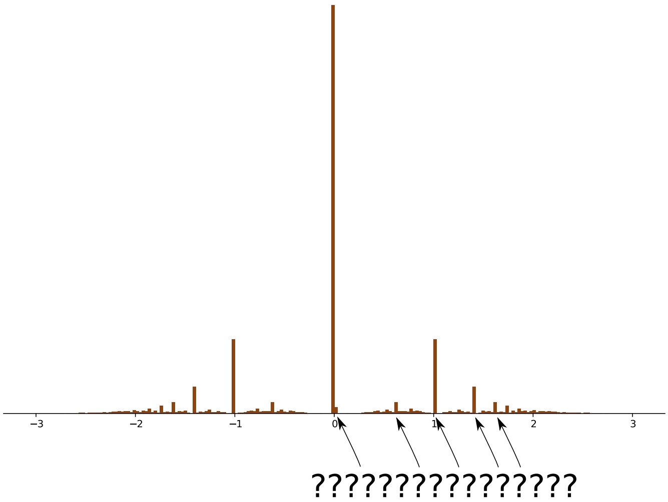
‘Histogram’ of μ_c with $c = 1$



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[Salez 2016]

T= some unimodular random tree with distribution ρ
$\mu_\rho = \mathbf{E}[\mu_{(T,o)}]$

$$\{\text{atoms of } \mu_\rho\} \subset \{\text{totally real algebraic integers }\} := \mathbb{A}$$

$\mathbb{A}$ = roots of polynomial P with integer coefficients, with only real roots.
$\mathbb{A}$ is dense in $\mathbb{R}$.

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Consequences:

- atoms of $\mu_c = \mathbb{A}$
- atoms of $\mu_{\text{skel}} = \mathbb{A}$

What happens **around** zero ?

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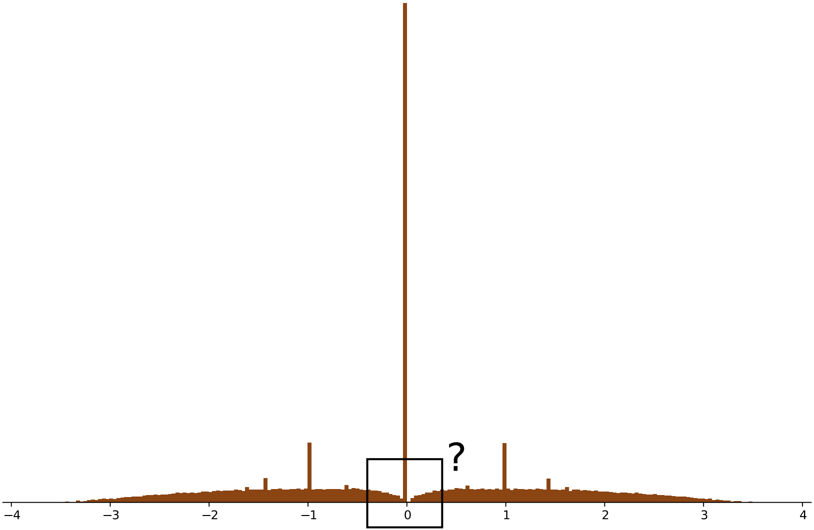
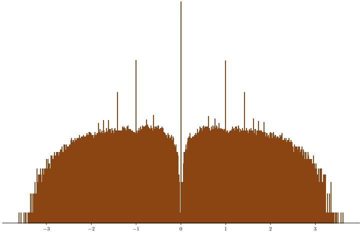
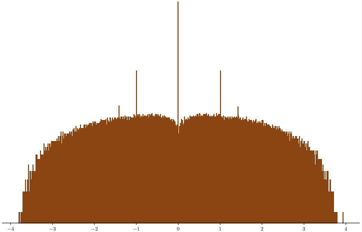


Extra simulations in log scale



$c = 2$



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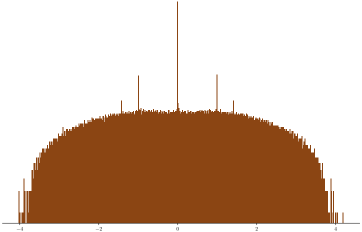
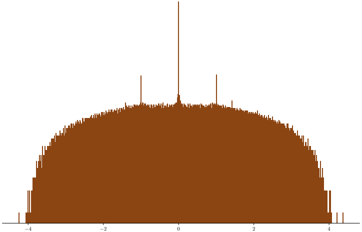


$c = 2,8$



$c = 3$

What happens at zero ?

Definition: we say that a measure μ has extended states at E if

$$\lim_{\varepsilon \to 0} \frac{\mu((E-\varepsilon, E+\varepsilon)) - \mu(\{E\})}{2\varepsilon} > 0$$

[C, Salez, 2018]

μ_c has extended states at zero

$$\Leftrightarrow$$

$$c > \mathrm{e}$$

+ easy generalization [C, 19+]: μ_{skel} has no extended states at zero.

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- (for Evgeny Spodarev) What is the Benjamini-Schramm limit of uniform fullerenes ?

Спасибо за внимание!