Applications of the perturbation formula for Poisson processes to elementary and geometric probability

Günter Last[1] Sergei Zuyev[2]

[1]Karlsruhe Institute of Technology, Germany
[2]Chalmers University of Technology, Sweden

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Bernoulli fields

Let $X = (X_1, \dots, X_n)$, $n \in \mathbb{N}$, are independent Bern(p) r.v.'s on $\Omega = \{0,1\}^n$. Let $X_{(i)}$ (resp. $X^{(i)}$) be a vector whose entries coincide with those of X except at the i-th coordinate, where the entry is 0 (resp. 1). For an event $A \subseteq \Omega$ denote

$$N_A^+ := \sum_{i=1}^n \mathbb{1}\{X^{(i)} \in A, X_{(i)} \notin A\},$$

$$N_A^- := \sum_{i=1}^n \mathbb{1}\{X_{(i)} \in A, X^{(i)} \notin A\}.$$

Definition

The coordinates i which contribute non-zero terms to N_A^+ (resp., to N_A^-) are called $(+)$-pivotal (resp., $(-)$-pivotal) for even A.

Variation formula

Margulis–Russo formula

$$\frac{d}{dp}\mathbf{P}_p(A) = \mathbf{E}_p[N_A^+ - N_A^-], \qquad (1)$$

where $\mathbf{E}_p$ denotes expectation with respect to the distribution $\mathbf{P}_p$ of X.

Its power is that it relates the probability of event to the geometry of the paths realising it. Many results in Percolation theory are obtained using it.

Binomial distribution

Let $S_n = X_1 + \ldots + X_n$ and $A = \{S_n \geq k\}$. Then

$$N_A^+ = \begin{cases} n-k+1, & \text{if } S_n = k-1, \\ k, & \text{if } S_n = k, \\ 0, & \text{otherwise.} \end{cases}$$

Since A is increasing event, $N_A^- = 0$ and

$$\frac{d}{dp}\mathbf{P}_p(A) = \mathbf{E}_p N_A^+ = (n-k+1)\mathbf{P}_p\{S_n = k-1\} + k\mathbf{P}_p\{S_n = k\}$$

$$= \frac{n!}{(k-1)!(n-k)!} p^{k-1}(1-p)^{n-k}.$$

Integral representation

Since $\mathbf{P}_0(A) = 0$,

$$\mathbf{P}_p\{S_n \geq k\} = \frac{n!}{(k-1)!(n-k)!} \int_0^p t^{k-1}(1-t)^{n-k} dt, \quad k \in \{1, \dots, n\}.$$

Similarly, if Z_n follows the Negative Binomial $\mathrm{NB}(r,p)$ distribution,

$$\mathbf{P}_p\{Z_n \geq k\} = \frac{(k+r-1)!}{(k-1)!(r-1)!} \int_0^p t^{r-1}(1-t)^{k-1} dt, \quad k \in \mathbb{N}.$$

Poisson process

Let λ be a fixed σ-finite measure on some measurable space $\mathbb{X}$ and $\theta \geq 0$. Consider a Poisson point process $\eta \sim \mathrm{PPP}(\theta\lambda)$ on $\mathbb{X}$ with intensity measure $\theta\lambda$. The corresponding distribution and expectation are denoted by $\mathbf{P}_\theta$ and $\mathbf{E}_\theta$. If A is a cylinder event, then (1) holds with

$$N_A^+ := \int \mathbb{1}\{\eta + \delta_z \in A, \eta \notin A\}\lambda(dz),$$
$$N_A^- := \int \mathbb{1}\{\eta \in A, \eta + \delta_z \notin A\}\lambda(dz),$$

Margulis–Russo analogue for PPP

By Mecke formula,

$$\mathbf{E}_\theta\, N_A^+ = \frac{1}{\theta}\,\mathbf{E}_\theta \int \mathbb{1}\{\eta \in A, \eta - \delta_z \notin A\}\eta(dz),$$

SZ'93

$$\frac{d}{d\theta}\mathbf{P}_\theta(A) = \frac{1}{\theta}\,\mathbf{E}_\theta \int \mathbb{1}\{\eta \in A, \eta - \delta_z \notin A\}\eta(dz) - \mathbf{E}_\theta \int \mathbb{1}\{\eta \in A, \eta + \delta_z \notin A\}\lambda(dz). \quad (2)$$

Pivotality

So, analogously to the Bernoulli case, the process points $z_i \in \eta$ such that $\eta \in A$, but $\eta - \delta_{z_i} \notin A$ maybe called pivotal points, whereas $z \in \mathbb{X}$ such that $\eta \in A$, but $\eta + \delta_z \notin A$ are called pivotal locations.

Formula (2) is a particular case of variation formula for $\mathbf{E}_\theta\, g(\eta)$, when the functional $g = \mathbb{1}_A$.

Difference operators

Let **N** be the set of configurations and $g : \mathbf{N} \mapsto \mathbb{R}$ be a measurable mapping. For $z \in \mathbb{X}$, introduce the difference operator $g \mapsto D_z g$:

$$D_z g(\varphi) = g(\varphi + \delta_z) - g(\varphi)$$

and its iterations:

$$D^k_{z_1,\dots,z_k} g = D_{z_k} D^{k-1}_{z_1,\dots,z_{k-1}} g \quad (z_1, \dots, z_k) \in \mathbb{X}^k.$$

Variation formula

Given any σ-finite measure ρ on $\mathbb{X}$, denote $\eta_\rho \sim \text{PPP}(\rho)$.

G. Last'14

Let λ be a σ-finite and let ν be a finite measure on $\mathbb{X}$. Let $g\colon \mathbf{N} \to \mathbb{R}$ be a measurable function such that $\mathbf{E}\,|g(\eta_{\lambda+\nu})| < \infty$. Let $\theta \in (-\infty, 1]$ such that $\lambda + \theta\nu > 0$. Then

$$\mathbf{E}\, f(\eta_{\lambda+\theta\nu}) = \mathbf{E}\, f(\eta_\lambda) + \sum_{k=1}^{\infty} \frac{\theta^k}{k!} \int \mathbf{E}\, D^k_{x_1,\dots,x_k} f(\eta_\lambda)\, \nu^k(d(x_1,\dots,x_k)),$$

where the series converges absolutely.

Derivatives

If $\mathbf{E}\,|g(\eta_{\theta_0\lambda})| < \infty$ for some $\theta_0 > 0$, then for any $\theta < \theta_0$,

$$\frac{d^k}{d\theta^k}\,\mathbf{E}\,g(\eta_{\theta\lambda}) = \int \cdots \int \mathbf{E}\,D^k_{z_1,\dots,z_k} g(\eta_{\theta\lambda})\,\lambda(dz_1)\cdots\lambda(dz_k).$$

In particular,

$$\left.\frac{d}{d\theta}\right|_{\theta=1} \mathbf{E}\,g(\eta_{\theta\lambda}) = \int \mathbf{E}[g(\eta_\lambda + \delta_z) - g(\eta_\lambda)]\,\lambda(dz).$$

Quite often, $\mathbf{E}\,D_z g$ is easier to compute than $\mathbf{E}\,g$ because the influence to g of added δ_z may be local.

Warm-up: Poisson distribution

Let $\mathbb{X}$ be a one-point set and λ is a unit mass on it. Then $\eta_\theta \sim \text{Po}(\theta)$. Consider $A = \{\eta_\theta \geq k\}$. Since

$$\mathbb{1}_A(\eta + \delta_z) - \mathbb{1}_A(\eta) = \mathbb{1}\{\eta\{z\} = k - 1\},$$

then

$$\mathbf{P}\{\eta_\theta \geq k\} = \int_0^\theta \frac{d}{dt}\mathbf{P}\{\eta_t \geq k\} = \int_0^\theta \frac{t^{k-1}}{(k-1)!}e^{-t}dt.$$

Erland distribution

By similar consideration, for $\zeta \sim \mathrm{Er}(n, \theta) = \Gamma(n, \theta)$,

$$\mathbf{P}\{\zeta \geq k\} = \frac{x^n}{(n-1)!} \int_0^\theta t^{n-1} e^{-tx} dt, \quad x \geq 0.$$

Compound Poisson distribution

Let ξ_i are *i.i.d.* with distribution Q on $\mathbb{R}$ with $Q\{0\} = 0$ and $Z_\theta = \sum_{k=1}^{\nu} \xi_i$, where $\nu \sim \text{Po}(\theta)$. Then $Z \sim \text{CPo}(\theta, Q)$, let $F(\theta, Q; x)$ be its *c.d.f.*

Take $\mathbb{X} := \mathbb{R}$ and $\eta \sim \text{PPP}(Q)$. Then $Z_\theta \stackrel{D}{=} \int z\,\eta(dz)$. Consider the event $A := \{Z_\theta \le x\},\ x \in \mathbb{R}$. Then, for $z \in \mathbb{R}$,

$$\mathbb{1}_A(\eta + \delta_z) - \mathbb{1}_A(\eta) = \mathbb{1}\{Z_\theta > x, Z_\theta + z \le x\} - \mathbb{1}\{Z_\theta \le x, Z_\theta + z > x\};$$

$$\frac{d}{d\theta}\mathbf{P}_\theta(A) = \mathbf{E}_\theta \int_{\mathbb{R}\setminus\{0\}} \mathbb{1}\{Z_\theta + z \le x\}\, Q(dz) - \mathbf{P}_\theta(Z_\theta \le x).$$

$$\frac{d}{d\theta}F(\theta, Q; x) = \int F(\theta, Q; x - z)\, Q(dz) - F(\theta, Q; x).$$

Strictly α-stable laws

Definition

A random vector ξ (or its distribution) is called strictly α-stable (StαS), if the following equality in distribution holds:

$$t^{1/\alpha}\xi' + (1-t)^{1/\alpha}\xi'' \overset{D}{=} \xi \quad 0 \le t \le 1,$$

where ξ', ξ'' are independent distributional copies of ξ.

In Euclidean spaces StαS laws exist only for $0 < \alpha \le 2$ and $\alpha = 2$ corresponds to the Gaussian distribution centred at the origin.

LePage representation

Symmetrical StαS random vectors in $\mathbb{R}^n$ with $\alpha < 2$ and all StαS random vectors with $\alpha < 1$ admit the following LePage series representation:

$$\xi := \xi_\theta \overset{D}{=} \int u\, \eta_\theta(du), \tag{3}$$

where $\eta_\theta \sim \text{PPP}(\Lambda_\theta)$, where

$$\Lambda_\theta := \theta \int_{\mathbb{S}^{n-1}} \int_0^\infty \mathbb{1}\{t^{-1/\alpha} u \in \cdot\}\, dt\, \hat{\sigma}(du)$$

is the Lévy measure on $\mathbb{R}^n \setminus \{0\}$ with $\sigma = \theta\hat{\sigma}$ on the sphere $\mathbb{S}^{n-1}$ called the spectral measure.

Thus the radial component of η_θ follows PPP with intensity measure $\theta\mu_\alpha$ with $\mu_\alpha[x, +\infty) = x^{-1/\alpha}$ and the angular component follows the distribution $\hat{\sigma}$.

Let S_σ be the support of the spectral measure σ. The corresponding stable law is non-degenerate if

$$K := \text{cone}(S_\sigma) = \{x \in \mathbb{R}^n : \ |x| > 0, \ x/|x| \in S_\sigma\}$$

has a positive n-volume. It is known that non-degenerate stable laws possess an infinitely differentiable density in its interior.

Density equations in $\mathbb{R}^n$

(i) The density f_θ of ξ_θ satisfies

$$nf_\theta(x)+\langle x, \nabla f_\theta(x)\rangle = \alpha \int [f_\theta(x)-f_\theta(x-z)]\,\Lambda_\theta(dz), \quad x \in \mathrm{Int}(K),$$

where $\langle\cdot\,,\cdot\rangle$ is the scalar product in $\mathbb{R}^n$.

(ii) Let $f_{|\xi_\theta|}$ denote the *p.d.f.* of the radius vector $|\xi_\theta|$. Then for all $r > 0$,

$$rf_{|\xi_\theta|}(r) = \alpha \int \Big[\mathbf{P}\big(|\xi_\theta| \le r\big) - \mathbf{P}\big(|\xi_\theta + z| \le r\big)\Big]\Lambda_\theta(dz).$$

Density equations in $\mathbb{R}_+$

The *c.d.f.* F_θ and the *p.d.f.* f_θ of a positive StαS on $\mathbb{R}_+$ with $0 < \alpha < 1$ are related through

$$f_\theta(x) + x f_\theta'(x) = \alpha^2\theta \int_0^x [f_\theta(x) - f_\theta(x-z)] z^{-\alpha-1}\, dz;$$

$$x f_\theta(x) = \theta\alpha^2 \int_0^x \left[F_\theta(x) - F_\theta(x-z)\right] z^{-\alpha-1}\, dz \quad \text{for all } x > 0,$$

Outline of the proof

Similarly to CPo, write

$$\frac{d}{d\theta}\mathbf{P}(\xi_\theta \in B) = \int \left[\mathbf{P}(\xi_\theta + z \in B) - \mathbf{P}(\xi_\theta \in B)\right] \Lambda_1(dz)$$
$$= \frac{1}{\theta} \int [\mathbf{P}(\xi_\theta \in B - z) - \mathbf{P}(\xi_\theta \in B)] \Lambda_\theta(dz)$$

and use the scaling $\xi_\theta \overset{D}{=} \theta^{1/\alpha}\xi_1$, so that the density and its gradient satisfy

$$f_\theta(x) = \theta^{-d/\alpha} f_1(\theta^{-1/\alpha}x),$$
$$\nabla f_\theta(x) = \theta^{-(n+1)/\alpha} \nabla f_1(\theta^{-1/\alpha}x).$$

Crofton's derivative formula

Consider m points uniformly and independently distributed in a finite volume $K \subset \mathbb{R}^n$ (Binomial point process). Assume we want to compute the probability P that these points satisfy certain property, *e. g.* the probability that the convex hull of $m = 4$ points is a triangle. Now expand monotonely the domain K to $K_t \supset K$ with $\cap_{t>0} K_t = K$. The Crofton's derivative formula relates the new probability P_t to satisfy the property when the points are now distributed in a larger domain K_t when $t \downarrow 0$.

Intuitively, the difference in P_t and P is due to: 1) the new scale factor due to the increase of volume of the domain; and 2) new possible configurations with points in $K_t \setminus K$. In the first order approximation, only one point in $K_t \setminus K$ matters. Its distribution should depend on the exact form of the expansion of K.

Settings

We consider a compact set K and $K_t = K + b(0,t)$ – the t-parallel set of $K \subset \mathbb{R}^d$.

Let $h\colon \mathbb{R}^n \to [0,\infty)$ be a continuous function and let λ be the measure on $\mathbb{R}^n$ with Lebesgue density h.

For $t \geq 0$ let λ_t be the restriction of λ to K_t and η_t be a Poisson process with intensity measure λ_t. Let $\mathcal{H}^{n-1}$ denote the Hausdorff measure.

Crofton formula for Poisson functionals

Assume, for simplicity, that K is a body, *i. e.* $\mathrm{cl}\,\overset{\circ}{K} = K$.

If $g(\eta_t + \delta_x)$ is continuous in $x \in K_{t_0}$ for some $t_0 > 0$, and there exists $c > 0$ such that

$$\left| \mathbf{E}\, D^k_{x_1,\dots,x_k} g(\eta_t) \right| \le c^k, \quad x_1, \dots, x_k \in K_{t_0},\ t \le t_0,\ k \in \mathbb{N}.$$

then for all $0 < t < t_0$

$$\frac{d}{dt} \mathbf{E}\, g(\eta_t) = \int_{\partial K_t} \mathbf{E}\left[g(\eta_t + \delta_x) - g(\eta_t) \right] h(x)\, \mathcal{H}^{n-1}(dx).$$

Under additional technical assumptions, this is also true for $K_0 = K$.

Crofton formula for Binomial process

Consider a *binomial processes* $\mathrm{BPP}(m, \lambda_t)$

$$\xi_t^{(m)} = \delta_{X_1} + \cdots + \delta_{X_m},$$

where $X_i \sim \lambda_t / \lambda_t(K_t)$ are *i.i.d.* r.v.'s in $\mathbb{R}^n$.

If g is bounded and $x \mapsto \mathbf{E}\, g(\xi_t^{(m-1)} + \delta_x)$ is continuous on K_{t_0} for each $t < t_0$, then

$$\frac{d}{dt} \mathbf{E}\, g(\xi_t^{(m)}) = \frac{m}{\lambda(K_t)} \int_{\partial K_t} \mathbf{E}\left[g(\xi_t^{(m-1)} + \delta_x) - g(\xi_t^{(m)})\right] h(x)\, \mathcal{H}^{n-1}(dx).$$

The proof uses the generalisation of the Steiner formula to non-convex sets [Hug, Last, Weil'04]. For bodies, the last theorem follows from [Baddeley'77], but we can also covers general closed sets. It this case, the integral above is over the set of $\partial^1 K$ of boundary points which have a unique outward 'normal' in the positive reach sence plus twice the integral over $\partial^2 K$ that have two normals.

References

- G. Last and S. Zuyev Applications of the perturbation formula for Poisson processes to elementary and geometric probability. `http://arxiv.org/abs/1907.09552` (2019)
- S. Zuyev. Russo's Formula for the Poisson Point Processes and its Applications, *Discrete Math. and Applications*, **3**, 63-73, (1993)
- G. Last. Perturbation analysis of Poisson processes. *Bernoulli*, 20(2):486–513, 2014.
- D. Hug, G. Last and W. Weil. A local Steiner-type formula for general closed sets and applications. Math. Z., **246**, 237–272, 2004.
- A. Baddeley. Integrals on a moving manifold and geometrical probability. Adv. Appl. Prob., **9**, 588–603, 1977.

Questions?